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Polynomial degeneracy for the first *m* energy levels of the antiferromagnetic Ising model

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Abstract. In this work, we continue our investigation on the antiferromagnetic Ising model on triangulations of closed Riemann surfaces. On the one hand, according to R. Moessner and A. P. Ramirez [11], the antiferromagnetic Ising model on triangulations exhibits geometrical frustration, a well-studied concept in condensed matter physics. Typical geometrically frustrated systems present an exponential ground state degeneracy. On the other hand, the dual graph of a triangulation of a closed Riemann surface is a cubic graph. Cubic bridgeless graphs have exponentially many perfect matchings [3, 5], which implies in the case of planar triangulations, an exponential ground state degeneracy. However, this phenomenon does not persist for triangulations of higher genus surfaces.

A possible explanation for a geometrically frustrated system with a low ground state degeneracy is that exponentially many states exist at a low energy level. In this work, we constructively show that this explanation does not match with the behavior of all triangulations of closed Riemann surfaces. To be more specific, for each integer $m \geq 1$, we construct a collection of triangulations $\{T_n\}_{n>N(m)}$ of a fixed closed Riemann surface with the property that the degeneracy of each of the first m energy levels of T_n is a polynomial in the order n of T_n.

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1. Introduction

In this work, all graphs considered are simple; namely, loopless and without multiple edges. A *triangulation* T of a closed Riemann surface S is an embedding of a graph (V, E) in S such that each facial boundary is a 3-cycle of (V, E) and each edge belongs to exactly two facial boundaries. Triangulations are fundamental objects of study in topological graph theory [12], computer science [13], and quantum geometry and statistical physics [1, 2], among others.

Let $T = (V, E, F)$ be a triangulation of a closed Riemann surface, where F is the set of facial boundaries given by the embedding of (V, E) in the surface. In the Ising model, values +1 and -1 are typically called *spins* and a *spin-assignment* on $U \subseteq V$ is a function $\sigma \colon U \to \{+1, -1\}$. A *state* on T is a spin-assignment on $U = V$. For every state σ on T, the energy of σ in the *antiferromagnetic* Ising model is given by the following expression

$$H(\sigma) = \sum_{uv \in E} \sigma_u \sigma_v. \tag{1}$$

States that provide the lowest possible energy are known as *ground states* and the number of distinct pairs σ, $-\sigma$ of ground states is called *ground state degeneracy*. The ground state degeneracy is a vastly studied parameter [6, 10]. In general, its asymptotic behavior determines the entropy of the system, which helps to comprehend physical phenomena associated with order and stability [15, 16].

In this work, we refer to an edge $e = uv \in E$ as *frustrated* by a spin-assignment σ on $U \subseteq V$, with $u, v \in U$, if $\sigma(u) = \sigma(v)$. Note that a given state σ on T is a ground state if and only if it frustrates the lowest possible number of edges. *Geometrical frustration* is a very broad and important concept from condensed matter physics [4, 11, 14]. A general definition is that a system exhibits geometrical frustration when a certain type of local order condition cannot be propagated through the system [14]. According to R. Moessner and A.P. Ramirez [11], the antiferromagnetic Ising model on triangulations T is a *geometrically frustrated system* because every state frustrates at least one edge of each face boundary of T. Further, the ground state degeneracy in a system that exhibits geometrical frustration is predicted to be very large in the size of the system [4, 11]. In fact, ground state degeneracy of the antiferromagnetic Ising model on triangulations of the plane is exponential, since it is equal to the number of perfect matchings of cubic bridgeless planar graphs [3, 8, 9]. Though the number of perfect matchings of general cubic bridgeless graphs is exponential [5], we have shown [7] that, in general, ground state degeneracy of the antiferromagnetic Ising model on triangulations does not have an exponential growth. In a personal communication, M. Mezzard explained to us that there might be geometrically frustrated systems with a non-highly degenerate ground state, in which case one would expect that, there are exponentially many states which provide low energy. In this work, we prove that triangulations of closed Riemann surfaces do not behave as expected in a very strong sense. We make this formal in the next paragraphs.

An *energy level* of the antiferromagnetic Ising model on a triangulation T is any integer ω satisfying that there is a state σ on T such that $\omega = H(\sigma)$. For every integer $k \geq 0$, the *k-th energy level* is defined to be the (k)th smallest energy level. Moreover, a state σ is said to be at the k-th energy level if $H(\sigma)$ equals the k-th energy level. A ground state is at the 0-th energy level. The number of distinct pairs $\sigma, -\sigma$ of states at a specific energy level is known as the *degeneracy of the energy level*.

In this work, and in the context of the antiferromagnetic Ising model, we prove that for each integer $m \geq 1$ there exists a collection of triangulations of a fixed closed Riemann surface such that, the degeneracy of each of its first m energy levels is a polynomial. More precisely, we establish the following.

Theorem 1. *For each $m \in \mathbb{N}$, there exist $n_0, g(m) \in \mathbb{N}$, and a collection of triangulations $\{\mathfrak{T}_{m,n}\}_{n>n_0}$ of the surface of genus $g(m) = O(m^{\log_2 3})$ such that the degeneracy of each of its first m energy levels is a polynomial in n, where n is the number of vertices of $\mathfrak{T}_{m,n}$.*

Theorem 1 completely generalizes our previous result [7], where we obtained the same conclusion for $m = 5$. In order to prove Theorem 1, in Section 2 we describe the construction of the collection $\{\mathfrak{T}_{m,n}\}_{n>n_0}$. Along this paper, we use some definitions and terminology introduced in [7], however, these are repeated here in order to make this document self-contained.

2. Construction of the collection $\{\mathfrak{T}_{m,n}\}_{n>n_0}$

The building blocks used to construct the triangulations of Theorem 1 are of three distinct types: (i) *root triangulations*, (ii) *inner triangulations*, and (iii) *leaf triangulations*. In order to describe them and establish their main properties, we need to introduce the notion of *punctured triangulations*.

Punctured triangulations. A Riemann surface with holes S' is a surface obtained from a closed Riemann surface S by removing a finite number of disjoint open discs; the boundaries of the open disc are not necessarily disjoint. A *punctured triangulation* of a Riemann surface with holes S' is an embedding of a graph in S' such that each face boundary is a 3-cycle of the graph and each hole of S' is circumscribed by a 3-cycle of the graph. On the one hand, if every hole in a punctured triangulation of S' is filled with an open disc, then we obtain a triangulation of S. On the other hand, a punctured triangulation can be naturally obtained from a triangulation by removing a set of faces. In general, each term defined for triangulations is naturally adapted to punctured triangulations. In Figure 2 this concept is illustrated.

The construction of the triangulations of Theorem 1 consists on two steps. Roughly speaking, in the first step we obtain a large punctured triangulation with exactly one hole by taking union of building blocks, which in turn are punctured toroidal triangulations. The description of each building block is in Subsection 2.1 and they are put together in Subsection 2.3.1. In the second step, we cover the hole of the large punctured triangulation obtained in the first step with a special type of planar triangulation referred to as *strip of triangles*, which is described in Subsection 2.2. The construction is completed in Subsection 2.3.2.

2.1. Description of the building blocks: root, inner and leaf triangulation. Each building block in our construction is a punctured triangulation obtained from the toroidal triangulation depicted in Figure 1. Though the building blocks in our construction do not differ much in structure, they perform very different tasks. While in the root triangulation is decided whether the current energy level has an exponential degeneracy or not, inner triangulations replicate the information encoded in the root triangulation and in the leaf triangulations the energy level is determined.

The names given to each of the building blocks are a natural consequence of the structure of the triangulations $\mathcal{T}_{m,n}$ of Theorem 1: each $\mathcal{T}_{m,n}$ can be seen as a rooted tree where the root triangulation is the root vertex, the inner triangulations are the inner vertices and the leaf triangulations are the leaves of the tree.

We now describe the three building blocks. We refer to the toroidal triangulation depicted in Figure 1 as *main triangulation* and use the labellings of its vertices as drawn in the same figure. Let us now agree on some terminology. In what follows, T' denotes a punctured triangulation obtained from the main triangulation by removing a subset of its faces, not necessarily non-empty. The subset of vertices $\{x, y, z\}$ of $V(T')$ is called *main triangle* of T', and the subsets of vertices $\{z, y, w_1\}, \{w_2, z, x\}, \{w_3, y, x\}$ of $V(T')$ are called *transfer triangles* of T'.

A *root triangulation* (and also, an *inner triangulation*) is a punctured triangulation obtained from the main triangulation by removing four of its faces: the face bounded by the main triangle and the three faces which are bounded by a transfer triangle. A *leaf triangulation* is a punctured triangulation obtained from the main triangulation by removing the face bounded by the main triangle. See Figure 2.

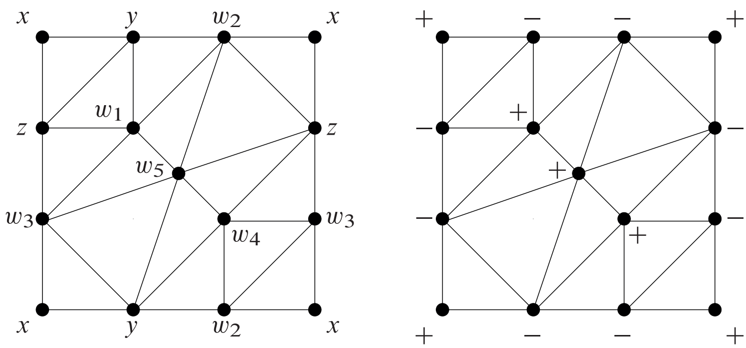

Figure 1. On the left hand side we find the main triangulation and the labeling of its vertices used throughout the paper. On the right hand side is depicted the standard spin-assignment on the vertices of the main triangulation.

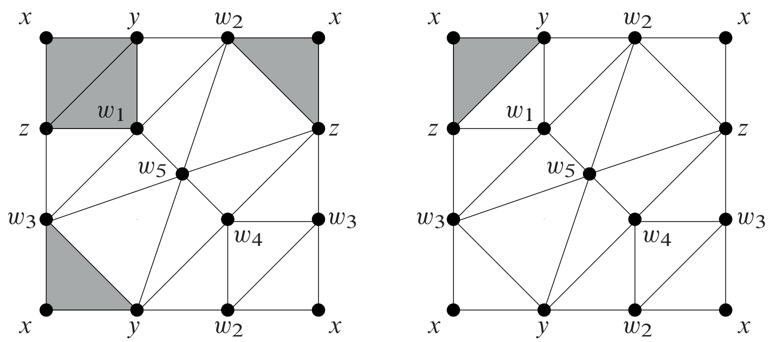

Figure 2. Root (and inner) triangulation is depicted in the left hand side. Leaf triangulation is depicted in the right hand side. Both triangulations are punctured triangulations obtained from the main triangulation. Shaded triangles correspond to removed triangles.

We define the *standard spin-assignment* on $V(T')$, and denote it by σ_s, as the unique spin-assignment on $V(T')$ which frustrates one edge of each face boundary of the main triangulation and $\sigma_s(x) = +1$. It is not hard to check that the spin-assignment depicted in Figure 1 is the *standard spin-assignment*. Further, the edges in $E(T')$ which are frustrated under σ_s are referred to as *link edges* of T'. By definition, every face boundary in the main triangulation has exactly one link edge.

A spin-assignment σ on $S \subseteq V(T')$ is called *congruent* on a subset of vertices $S' \subseteq S$ if $\sigma(S') \in \{\sigma_s(S'), -\sigma_s(S')\}$. Otherwise σ is called *incongruent* on S'. The following claim is trivial.

Claim 2. *A spin-assignment σ is incongruent on the main triangle $\{x, y, z\}$ of T' if and only if $\sigma(x) = \sigma(z)$ or $\sigma(x) = \sigma(y)$.*

We now examine some key properties of the building blocks. Along this work, we say that a face of a triangulation (or of a punctured triangulation whenever the face exists[1]) is *frustrated* by a spin-assignment σ if each edge of its boundary is frustrated by σ. The motivation to introduce the notion of frustrated faces is that, in our setting, the energy provided by a spin-assignment depends only on the number of frustrated and non-frustrated faces that it holds. More precisely, note that for a triangulation $T = (V, E, F)$ and a spin-assignment σ on $V(T)$, the energy $H(\sigma)$, defined in equation (1), is given by

$$H(\sigma) = \frac{3}{2}|\{f \in F \text{ frustrated by } \sigma\}| - \frac{1}{2}|\{f \in F \text{ not frustrated by } \sigma\}|. \tag{2}$$

Lemma 3. *Let T be a leaf triangulation and let $\{x, y, z\}$ be the main triangle of T with $\{z, y\}$ a link edge.*

(i) *If σ is a spin-assignment on $V(T)$ incongruent on $\{x, y, z\}$, then σ is incongruent on at least two transfer triangles of T and it frustrates at least one face of T.*

(ii) *If σ' is a spin-assignment on $\{x, y, z\}$ congruent on $\{x, y, z\}$, then there exists a unique spin-assignment σ on $V(T)$ so that $\sigma(\{x, y, z\}) = \sigma'(\{x, y, z\})$, σ frustrates no face of T and σ is congruent on each transfer triangle of T.*

Proof. We prove (i) first. Assume that $\sigma(x) = \sigma(z)$. The case $\sigma(x) = \sigma(y)$ follows analogously due to the symmetry of the leaf triangulation. The assumption $\sigma(x) = \sigma(z)$ implies that σ is incongruent on $\{w_2, z, x\}$. If $\sigma(y) = \sigma(x)$, then σ is incongruent on $\{w_3, y, x\}$ and if $\sigma(y) \neq \sigma(x)$, then σ is incongruent on $\{z, y, w_1\}$. Next, we prove that at least one face of T is frustrated by σ.

We can assume that $\sigma(w_2) \neq \sigma(x)$, otherwise $\{w_2, z, x\}$ is frustrated by σ. We suppose first $\sigma(y) = \sigma(x)$. If $\sigma(w_1) = \sigma(x)$ or $\sigma(w_3) = \sigma(x)$, then at least one face is frustrated by σ. Hence, we can assume $\sigma(w_1) = \sigma(w_3) \neq \sigma(x)$. If $\sigma(w_5) = \sigma(w_1)$, then at least two faces are frustrated by σ. Thus, we can further assume that $\sigma(w_5) \neq \sigma(w_1)$. Finally, $\sigma(w_4) = \sigma(w_3)$ or $\sigma(w_4) \neq \sigma(w_3)$, implies that at least one face is frustrated by σ. For the second case, we suppose $\sigma(y) \neq \sigma(x)$. We can assume that $\sigma(w_1) = \sigma(w_4) \neq \sigma(y)$, otherwise σ frustrates

[1] In a punctured triangulation removed faces do not count as faces of the embedding. Thus, for the same underlying graph and the same spin-assignment frustrated faces in a triangulation are not necessarily frustrated faces in a punctured triangulation.

at least one face. We can further assume $\sigma(w_3) \neq \sigma(z)$, otherwise at least one face is frustrated by σ. Finally, $\sigma(w_5) \in \{+, -\}$ leads to one face frustrated by σ.

In order to prove (ii), we first note that by definition of the standard spin-assignment, either σ_s or $-\sigma_s$ satisfies the required properties (see Figure 1). We now prove uniqueness. Let σ be the spin-assignment on $V(T)$ satisfying the required properties. Then $\sigma(w_1) \neq \sigma(z)$, $\sigma(w_2) = \sigma(z)$ and $\sigma(w_3) = \sigma(y)$ since σ is congruent on the transfer triangles of T. Moreover, if σ does not frustrate any face of T, then $\sigma(w_1) = \sigma(w_4) = \sigma(w_5)$. □

We observe that Lemma 3 also applies to root triangulations and inner triangulations, as these can be obtained from a leaf triangulation by removing the three faces which are bounded by a transfer triangle. More precisely, we have the following corollary for root and inner triangulations.

Corollary 4. *Let T be a root triangulation or an inner triangulation and let $\{x, y, z\}$ be the main triangle of T with $\{z, y\}$ a link edge.*

(i) *If σ is a spin-assignment on $V(T)$ incongruent on $\{x, y, z\}$, then σ is incongruent on at least two transfer triangles of T.*

(ii) *If σ' is a spin-assignment on $\{x, y, z\}$ congruent on $\{x, y, z\}$, then there exists a unique spin-assignment σ on $V(T)$ so that $\sigma(\{x, y, z\}) = \sigma'(\{x, y, z\})$, σ frustrates no face of T and σ is congruent on each transfer triangle of T.*

2.2. Strip of triangles. For every $k \geq 0$, the *strip of triangles* Δ_k on k inner vertices is the plane triangulation defined as follows. Let Δ_0 be a plane triangle on vertex set $\{x', y_0, z'\}$. For $i \geq 1$, let Δ_i be the plane triangulation obtained from Δ_{i-1} by first inserting a new vertex y_i into the inner face $\{x', y_{i-1}, z'\}$ of Δ_{i-1}, and then connecting the new vertex y_i to each vertex x', y_{i-1} and z' (see Figure 3). We say that x', z' are the *big vertices* of Δ_k. Strips of triangles belong to a family of plane triangulations usually known as stack triangulations (see [8]). The next statement describes the useful properties of the strips of triangles.

Proposition 5. *For every $k \geq 0$, let Δ_k be the strip of triangles with outer face $\{x', y_0, z'\}$ and big vertices x', z'. Define $\Phi_q(k)$ as the number of spin-assignments on $V(\Delta_k)$ that assign $\{+, -, +\}$ to $\{x', y_0, z'\}$ and frustrate exactly $q \geq 0$ faces of Δ_k. Then $\Phi_q(k) \in O(k^q)$.*

Proof. We proceed by induction on q. For the case $q = 0$ is easy to check that if a spin-assignment σ on $\{x', y_0, z'\}$ assigns $\{+, -, +\}$ to $\{x', y_0, z'\}$, then there is a unique way to extend σ to a spin-assignment on $V(\Delta_k)$ so that no face is

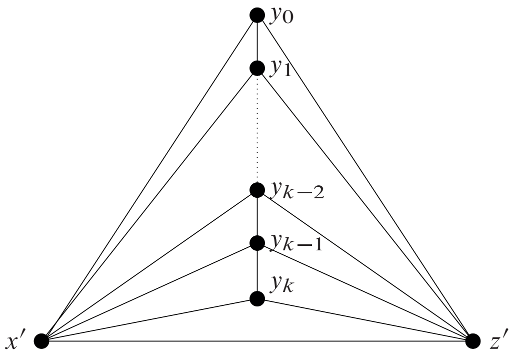

Figure 3. Strip of triangles.

frustrated. Thus, $\Phi_0(k) = 1$ for every $k \geq 0$. Note that any spin-assignment on $V(\Delta_k)$ that assigns $\{+, -, +\}$ to $\{x', y_0, z'\}$ frustrates at most k faces of Δ_k, and then, $\Phi_q(k) = 0$ whenever $q > k$. For the case that $q \leq k$, it is easy to check that

$$\Phi_q(k) = \Phi_{q-1}(k-1) + \Phi_q(k-1) \tag{3}$$

which together with the induction hypothesis, $\Phi_{q-1}(k-1) \in O(k^{q-1})$, implies that $\Phi_q(k) \in O(k^q)$. □

2.3. Assembling the building blocks. In order to define the collection of triangulations $\{\mathcal{T}_{m,n}\}_{n>n_0}$ for each $m \in \mathbb{N}$, we first recursively define a collection of punctured triangulations $\{\mathcal{T}_m\}_{m\in\mathbb{N}}$, with $\mathcal{T}_m$ of genus $g(m)$ and exactly one hole. Each $\mathcal{T}_m$ is a union of building blocks.

2.3.1. Recursive construction of $\{\mathcal{T}_m\}_{m\in\mathbb{N}}$. Let $g(m)$ denote the genus of $\mathcal{T}_m$ for each $m \in \mathbb{N}$. We first describe $\mathcal{T}_0$. Triangulation $\mathcal{T}_0$ is a leaf triangulation, $g(0) = 1$ and $|V(\mathcal{T}_0)| = 8$. We now suppose $m \geq 1$. If m is odd, then the triangulation $\mathcal{T}_m$ is the union of a root triangulation and three copies of $\mathcal{T}_{(m-1)/2}$: each of the transfer triangles of the root triangulation is identified with the main triangle of a copy of $\mathcal{T}_{(m-1)/2}$ so that link edges coincide. Hence, $\mathcal{T}_m$ has exactly one hole, namely the hole bounded by the main triangle of the the root triangulation used in its construction. Moreover, $g(m) = 1 + 3g((m-1)/2)$ and $|V(\mathcal{T}_m)| = 8 + 3\cdot(|V(\mathcal{T}_{(m-1)/2})| - 3) = 3\cdot|V(\mathcal{T}_{(m-1)/2})| - 1$. If m is even, then the triangulation $\mathcal{T}_m$ is the union of a root triangulation, two copies of $\mathcal{T}_{(m/2)}$ and a copy of $\mathcal{T}_{(m/2)-1}$: two transfer triangles of the root triangulation are identified with the two main triangles of the two copies of $\mathcal{T}_{(m/2)}$ and the third transfer triangle is identified with the main triangle of the copy of $\mathcal{T}_{(m/2)-1}$. In both cases the identification is made so that link edges coincide. As before, $\mathcal{T}_m$ has exactly one hole. Moreover, $g(m) = 1 + 2g(m/2) + g((m/2) - 1)$ and $|V(\mathcal{T}_m)| = 8 + 2 \cdot (|V(\mathcal{T}_{(m/2)})| - 3) + |V(\mathcal{T}_{(m/2)-1})| - 3 = 2 \cdot |V(\mathcal{T}_{(m/2)})| + |V(\mathcal{T}_{(m/2)-1})| - 1$. Again, we refer to the main triangle of the root triangulation used in the construction of $\mathcal{T}_m$ as the *main triangle of* $\mathcal{T}_m$. Clearly, there is a positive constant c such that $|V(\mathcal{T}_m)| \leq c \cdot m^2$ for each $m \geq 1$.

Lemma 6. *Let σ be a state of $\mathcal{T}_m$. If σ is incongruent on the main triangle then, the number of faces frustrated by σ on $\mathcal{T}_m$ is at least $m + 1$.*

Proof. By induction on m. If $m = 0$, then the result follows by Lemma 3(i). Suppose $m > 0$ is odd. Due to Corollary 4(i), the state σ is incongruent on at least two transfer faces of the root triangulation used in the construction of $\mathcal{T}_m$. By construction of $\mathcal{T}_m$, these two transfer faces are main triangles of two copies of $\mathcal{T}_{(m-1)/2}$. By the induction hypothesis, the number of faces frustrated by σ on each copy of $\mathcal{T}_{(m-1)/2}$ is at least $\frac{m-1}{2} + 1$ and the result follows. Now, suppose $m > 1$ is even. Again, due to Corollary 4(i), the state σ is incongruent on at least two transfer faces of the root triangulation used in the construction of $\mathcal{T}_m$. By construction of $\mathcal{T}_m$, these two transfer faces are main triangles of either two copies of $\mathcal{T}_{(m/2)}$ or of a copy of $\mathcal{T}_{(m/2)-1}$ and a copy of $\mathcal{T}_{(m/2)}$. By the induction hypothesis, in the first case, we obtain that the number of faces frustrated by σ on $\mathcal{T}_m$ is at least $m + 2$ and in the second case, it is at least $m + 1$. □

2.3.2. Construction of $\mathcal{T}_{m,n}$. Let $\{\mathcal{T}_m\}_{m \in \mathbb{N}}$ be the collection of punctured triangulations defined in Subsection 2.3.1. Let $m \in \mathbb{N}$, $n > n_0 = |V(\mathcal{T}_m)|$ and $k = n - |V(\mathcal{T}_m)|$. Let Δ_k be the strip of triangles on k internal vertices, as described in Subsection 2.2, with outer face $\{x', y_0, z'\}$. We keep the labeling of the vertices in the main triangle of $\mathcal{T}_m$ as depicted in Figure 1 and x', z' are the big vertices of Δ_k as depicted in Figure 3. The triangulation $\mathcal{T}_{m,n}$ is the union of Δ_k and $\mathcal{T}_m$ following the rule: the main triangle $\{x, y, z\}$ of $\mathcal{T}_m$ is identified with the outer face $\{x', y_0, z'\}$ of Δ_k so that edge xz of the main triangle coincides with edge $x'z'$ of the outer face of Δ_k. The genus of $|V(\mathcal{T}_{m,n})|$ is $g(m)$ and $|V(\mathcal{T}_{m,n})| = k + |V(\mathcal{T}_m)| = n$.

We are now ready to prove Theorem 1.

3. Proof of Theorem 1

Let $m \in \mathbb{N}$ and $n_0 = |V(\mathcal{T}_m)| \leq c \cdot m^2$, where $\{\mathcal{T}_m\}_{m \in \mathbb{N}}$ is the collection of punctured triangulations defined in Subsection 2.3.1 and c is some positive constant. We show that the collection of triangulations $\{\mathcal{T}_{m,n}\}_{n>n_0}$ defined in Subsection 2.3.2 proves Theorem 1. By construction, the genus of each triangulation $\mathcal{T}_{m,n}$, denoted by $g(m)$, is described according to the following recursively defined function $g(0) = 1$ and for $m > 0$

$$g(m) = \begin{cases} 1 + 2g\left(\frac{m}{2}\right) + g\left(\frac{m}{2} - 1\right) & \text{if } m \text{ is even,} \\ 1 + 3g\left(\frac{m-1}{2}\right) & \text{if } m \text{ is odd.} \end{cases}$$

and $|V(\mathcal{T}_{m,n})| = n$. We now prove that in $\mathcal{T}_{m,n}$ the degeneracy of each of its first m energy levels is a polynomial in n.

We use the following claim.

Claim 7. *For each $0 \leq t \leq m$, there is a spin-assignment on $V(\mathcal{T}_{m,n})$ which frustrates exactly t faces of $\mathcal{T}_{m,n}$.*

Proof. Let σ' be a congruent spin-assignment on the main triangle $\{x, y, z\}$ of $\mathcal{T}_{m,n}$. Due to Lemma 3(ii) and Corollary 4(ii), there is a spin-assignment σ on $\mathcal{T}_m$ such that $\sigma(\{x, y, z\}) = \sigma'(\{x, y, z\})$ and σ frustrates no face of $\mathcal{T}_m$. Further, the congruent spin assignment σ' on $\{x, y, z\}$ is a spin-assignment on the outer face of the strip of triangles Δ_k which assigns distinct spins to the big vertices and hence, due to Proposition 5, for each $0 \leq t \leq m$ there is a spin-assignment σ_t on Δ_k that frustrates exactly t faces of Δ_k. Thus, the spin-assignment $\sigma \cup \sigma_t$ on $V(\mathcal{T}_{m,n})$ proves the claim for each $0 \leq t \leq m$. □

Let $0 \leq t \leq m$ be fixed. We now give an upper bound on the degeneracy of the t-th energy level and show that it is a polynomial in n. Combining the previous claim and equation (2), we obtain that a spin-assignment σ on $V(\mathcal{T}_{m,n})$ is at the t-th energy level if and only if σ frustrates exactly t faces of $\mathcal{T}_{m,n}$. Therefore, due to Lemma 6, a spin-assignment σ at the t-th energy level must be congruent on the main triangle $\{x, y, z\}$ of $\mathcal{T}_{m,n}$ and hence, it assigns distinct spins to the big vertices of the outer face of the strip of triangles Δ_k. According to Proposition 5, the number of spin-assignments on Δ_k which frustrates exactly q faces of Δ_k and assigns distinct spins to the big vertices is $O(k^q)$ for each $q \leq t$. Since there exist $2^{|V(\mathcal{T}_m)|}$ spin-assignments on $\mathcal{T}_m$, the number of spin-assignments on $\mathcal{T}_{m,n}$ which frustrate exactly t faces of $\mathcal{T}_{m,n}$ is at most $2^{|V(\mathcal{T}_m)|} \sum_{q \leq t} O(k^q) \leq O(n^t)$.

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