Institute of Statistics

Inventory Models and their Implementation and Application

Thesis

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Miguel A. Parra.

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Objectives and organization of the thesis

Objectives

The objectives of this study are presented below.

Main objective

To implement and apply a methodology based on inventory models with the `R` software.

Specific objectives

1. To optimize contribution margins of food services by modeling independent component demand.

2. To explore the potential use of the Birnbaum-Saunders distribution in inventory management

3. To introduce inventory management for new products with triangularly distributed demand and lead-time.

3. To collect and apply R software packages which allow calculation, adjustment, and the estimation of distributional parameters for inventory models for perishable and nonperishable inventories, develop of an algorithm for goodness of fit through tests and graphics.

Organization of the thesis

This thesis is organized in three chapters. In Chapter 1, we optimize the contribution margins of food services by modeling independent component demand. In Chapter 2, we explore the potential use of the Birnbaum-Saunders distribution in inventory management. In Chapter 3, we introduce inventory management for new products with triangularly distributed demand and lead-time. Each chapter is self-contained and presents their notations and background, which do not necessarily correspond to those of other chapter. The conclusions and list of references are common to all of the chapters and presented at the end of document.

Chapter 1

Optimization of contribution margins of food services by modeling independent component demand

1.1 Introduction

Systems of supply and inventory policies affect positively logistics of companies, minimizing the involved costs, and reducing inefficiencies in their management. It is known that the total inventory cost is function of purchasing (PC), ordering (OC) and storing (SC) costs; see Hillier and Lieberman (2005). Several authors have discussed the importance of having optimal supply and inventory policies in a company and an efficient management of its logistics; see Blankley et al. (2008) and Kogan and Tell (2009). These aspects of logistics are also present in collective food service companies; see Ramirez (2013). Such services prepare portions of a food menu according to diverse specifications, including nutritional and sanitary issues, with respect to the different types of clients who consume this menu; see Marambio et al. (2005). The increase in amount of food services is generating an important source of employment in the countries and providing multiple market opportunities. This is attributed to the need that people have to eat outside of their homes due to activities related to businesses, factories, hospitals, schools and universities. Because of the service diversity, complexity of this food industry has grown considerably, requiring a professional management and a regulation from government agencies; see Marden (2004).

In Chile, collective food services are considered in the group of small and medium enterprises. Many of these Chilean services are not optimizing their supply of raw materials. These materials form the inventory assortment, which is divided in perishable (as fruits, meats, vegetables) and non-perishable products with greater storing capacity subject to shortage; see Grant et al. (2006). Logistics of these raw materials are based on the monthly planning of the menu, which are guided by nutritional considerations. However, this management can be improved by using inventory policies, which allows contribution margins (CMs) of the company to be increased; see Soman (2006) and Nicolau (2009) for a case study in hotels. CMs are the gross profits of a company and summarize the movements of income and costs, which may be direct (variable costing) and indirect (absorption costing). These margins vary depending on sold units, unit costs of the product, ratio between them and the total costs and fixed costs involved; see Ramanathan (2006).

An optimal inventory policy can be attained choosing the most adequate inventory model, whose decision involves several aspects; see Botter and Fortuin (2000), Braglia et al. (2004a), Braglia et al. (2004b), Wanke (2011) and Wanke (2008b). When non-perishable (multi-period) products are considered, inventory models are classified in two types: pull and push, which range from the economic order quantity (EOQ) to the just in time (JIT) supply; see Wanke (2008a). The EOQ model is the cornerstone of several software packages for inventory control and is widely used in practice; see Nahmias (2001). The JIT method is useful for raw materials that can be supplied as timely as they are required, although it imposes constraints to the logistics limiting its use for some types of products in food services; see Carter et al. (2000) and Wanke et al. (2008). Chiu (2010) discussed models for multi-period products where shortage is not permitted, seeking to find the EOQ and reorder point (ROP), which may be appropriate for groceries, often used by food services. Considering lead time (LT) in the modeling makes the assumptions of the model to be more adherent to real world settings; see Ben-Daya and Raouf (1994). The EOQ model is used altogether with the ROP in inventory control to determine safety stocks (SS) under both random LT and demand, which randomness directly affects the operation of a logistics system; see Speh and Wagenheim (1978) and Wanke (2008a). Perishable (single-period) products can only be stored during a limited period. These products usually correspond to fruits, meats and vegetables, which are essential raw materials in a food service. When these types of products are considered, the model based on the critical ratio (CR) or service level is often considered; see (Hillier and Lieberman, 2005, pp. 961-975).

Multiple and single period models must consider that the demanded quantity of a product cannot be predicted accurately due to several factors, making it to be a random variable (RV) and, therefore, its behavior should be described by a statistical distribution (or probabilistic model); see Johnson et al. (1995). The Gaussian (or normal) distribution is often used for describing data of three RVs involved in inventory models, which are demand, LT, and demand during LT. It is known that this distribution is validly used for RVs that take negative and positive values, so that quantities less than zero could be admitted in the modeling, which is not possible in practice for the three mentioned RVs; see Keaton (1995) and Nahmias (2001). Mentzer and Krishnan (1988) studied the non-normality effect on the inventory control, indicating that demand for products that present a normal distribution is found in few practical cases. This is because demand data often follow asymmetric distributions; see Moors and Strijbosch (1988). In any case, the normality assumption must be checked by goodness-of-fit methods; see Castro-Kuriss et al. (2009), Castro-Kuriss et al. (2010), Barros et al. (2014) and Castro-Kuriss et al. (2014). Thus, using the normal distribution to model the demand and LT and to determine the ROP and SS can provoke wrong results, leading to stock shortage or excess. Some non-normal distributions used for describing demand or LT in inventory models are the gamma or Erlang, inverse Gaussian (IG), lognormal (LN), uniform and Weibull; see Burgin (1975), Tadikamalla (1981), Lau (1989), Wanke (2008c) and Cobb et al. (2013).

A probability model with positive asymmetry that is receiving considerable attention is the Birnbaum-Saunders (BS) distribution; see (Johnson et al., 1995, pp. 651-663). This is due to its good properties and its relation with the normal distribution, which permits the BS distribution to be behaved as the LN distribution, but with properties that the LN does not have. Its applications range diverse fields including business and industry; see Jin and Kawczak (2003), Bhatti (2010), Ahmed et al. (2010), Leiva et al. (2011b), Vilca et al. (2010), Sanhueza et al. (2011), Villegas et al. (2011), Ferreira et al. (2012), Leiva et al. (2012), Paula et al. (2012), Leiva et al. (2014d), Marchant et al. (2013), Leiva et al. (2014a), Leiva et al. (2014c) and Leiva et al. (2014e). The BS distribution includes the duration of the counting period (daily or weekly), which can be changed without collecting extra data, among other interesting properties, allowing the BS distribution to be a good candidate for describing demand data; see Fox et al. (2008).

A good statistical modeling of the demand data and a scientific management of inventories for collective food service companies can maximize their CMs, implying a better competitiveness, efficiency and profitability of these companies. This can be helpful in making optimal decisions.

The main objective of this paper is to propose a methodology useful for food services that allows their CMs to be optimized. This methodology is based on statistical tools, inventory management models and financial indicators. Specifically, the methodology uses probabilistic models that describe the behavior of demand data for raw materials employed by food services that prepare a daily menu. Hereafter, we refer to these raw materials as components (or products) forming part of a food menu. Then, the logistics process is optimized by using inventory models that depends on the type of product from the corresponding assortment. Hence, the CMs of the company are measured by using absorption costing and improved by means of logistics management. Such an improvement must be reached when comparing the financial results obtained from the optimized system with respect to the non-optimized system used by the food service. Because some authors stressed the need to conduct case studies that reduces the gap between theory and practice and enables the researchers to increase their background Wagner and Lindemann (2008), we apply this methodology to the case study of a food company that serves the staff of a Chilean hospital.

This chapter is organized as follows. In Section 1.2, we propose our methodology. In Section 1.3, we conduct a case study for a Chilean food service. In Section 1.4, we provide an illustration for one product from the inventory assortment.

1.2 Methodology

In this section, we provide a methodology for food services that allows CMs to be optimized. First, we discuss assumptions and limitations of our methodology. Second, we mention how the demand data for components of a food menu should be collected. Third, we present the statistical tools needed to fit a demand data set to a suitable distribution. Fourth, we detail the inventory management models to be used for optimizing the supply system based on the selected distributions. Fifth, we describe the financial indicators of our methodology,financial indices are introduced in order to calculate the costs of the exercise and replaces the expression 3.14 (total cost equation exercise). An algorithm that summarizes this methodology is provided.

1.2.1 Assumptions and limitations

The main assumptions of our methodology are (i) random demand, (ii) demand time series free of seasonality and trend, (iii) independent component demand, (iv) constant LT and (v) the need to ascertain managerial costing calculations. Some limitations of our methodology are related to (i) additional research needed to improve the results, especially introducing aspects to be more adherent to real world settings, such as issues related to seasonality, trend and independence, and (ii) relevant costs of the operation characteristics. Another of the limitations is that demand is intermittent and not present in the period LT (timeout) or is less than 2 daily demands therefore the most appropriate method is the JIT (just in time).If demand is intermittent this means that fewer than estimated by the EOQ model, the model assumes daily sales throughout the exercise period (Investment in idle supercharged model).

Note that shortage costs for non-perishable products could be unavailable and then they could not be incorporated in the analysis. Unlike situations proposals by Silver et al. (1998) and Zipkin (2000), in our methodology, there is no CMs or penalties that are imposed to a product with unsatisfied demand. In practice, for food companies, a product is replaced by another when it is missing and the customer continues making his(her) meals. We set a target level of service based on a safety factor (SF), instead of the simultaneous optimization of the EOQ and SF. You can see that the EOQ model with SS's, generates larger amounts to those seen in other models to prevent shortage costs, also generating reorder points up to three times the hope of demand (for cases of gaussian or average distributions), recommended is to apply this model to companies with a high turnover of products such as a multinational corporation. This is due to the eminently practical character of our study, which objective is, among others, to transfer knowledge and management of inventory policy over time to the studied company. Using the SF in an inventory policy necessarily requires the manager to think in terms of service level and inventory segmentation by levels of criticality with respect to shortage of items. When the simultaneous optimization of EOQ and SF is carried out, these issues are less explicit for the manager.

1.2.2 Recording the data

We recommend to design and implement a record system for all the products that form the inventory assortment of raw materials of the food service company. This system must be based on identification code, unit PC, demanded quantity, price, date and time of entry and exit of products used in the preparation of food portions; see Harvey (2002) and Yajiong (2008). The record system must be carried out via individual identification using bar codes and developed for demand profiles of the products of the inventory assortment, during the time period planned for the study. We recommend a period of six months (26 weeks). Initially, we considered 26 weeks (half year) as a sample of convenience based on the project budget for data collection. However, we could collect data for one week more, so that this additional week was considered to increase the sample size.

1.2.3 Demand statistical distributions

Based on the record system mentioned, the demand data needed to model the distribution of the demanded quantity for each component must be collected and then the demand distribution fitted. Until very recently, one of the problems for using a demand distribution different from the normal model was the limitation of statistical software. However, today this is not a problem, first because at present we have a number of statistical software that has implemented several statistical distributions and, second, the scientific community has at its disposal a non-commercial and open source software for statistics and graphs, named R, which can be obtained at no cost from www.r-project.org. The statistical software R is being currently very popular in the international scientific community. Then, to perform a statistical analysis of demand data, we use the R software and also employ some software packages to carry out more specific statistical analysis. As mentioned, the gamma, IG, LN, uniform and Weibull distributions have been used for modeling the demand or the LT in inventory problems and they are implemented in R software packages named gamlss and ig; see Stasinopoulos and Rigby (2007) and Leiva et al. (2008),One package mentioned in this thesis is the gamlss package for the mle alternatively we also have the MASS package with the "fitdistr" function which is quite helpful; see Shapiro et al. (2002). Statistical analysis based on BS distributions, including a version known as the BS-Student-t (BS-t) distribution, which has been proven to provide robust estimates of its parameters against outliers Paula et al. (2012), can be conducted by means of an R software package named gbs; see Barros et al. (2009). Next, we provide some useful results for all of these distributions; see details in Johnson et al. (1995).

The BS distribution A RV D following the BS distribution with shape $\alpha > 0$ and scale $\beta > 0$ parameters is denoted by $D \sim \text{BS}(\alpha, \beta)$, where "$\sim$" means "distributed as". In this case, the probability density (PDF) and cumulative distribution (CDF) functions of D are respectively

$$\begin{aligned} f_D(d) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2\alpha^2}\xi^2(d/\beta)\right) \frac{[d/\beta]^{-1/2} + [d/\beta]^{-3/2}}{2\alpha\beta} \quad \text{and} \\ F_D(d) &= \Phi\left([1/\alpha]\xi(d/\beta)\right), \quad d > 0, \end{aligned}$$

where $\xi(y) = \sqrt{y} - \sqrt{1/y}$ and $\Phi(\cdot)$ is the standard normal CDF. The corresponding quantile function (QF) is $d(q) = F_D^{-1}(q) = \beta[\alpha z(q)/2 + \sqrt{(\alpha z(q)/2)^2 + 1}]^2$, for $0 < q < 1$, where $z(q)$ is the standard normal or N(0, 1) QF and $F_D^{-1}(\cdot)$ is the inverse CDF. Note that $d(0.5) = \beta$, that is, β is also the median or 50th percentile of the distribution. The mean and variance of D are $\text{E}[D] = \beta[1+\alpha^2/2]$ and $\text{Var}[D] = \beta^2\alpha^2[1+5\alpha^2/4]$. In addition, BS RVs ($D$) and standard normal ($Z$) are related by $D = \beta[\alpha Z/2 + \sqrt{(\alpha Z/2)^2 + 1}]^2 \sim \text{BS}(\alpha, \beta)$ and $Z = [1/\alpha]\xi(D/\beta) \sim \text{N}(0, 1)$. Also, $W = Z^2$ follows a chi-squared distribution with one degree of freedom, which is useful for goodness of fit. The BS distribution holds the scale and reciprocal properties, that is, $c\,D \sim \text{BS}(\alpha, c\,\beta)$, with $c > 0$, and $1/D \sim \text{BS}\,(\alpha, 1/\beta)$, respectively.

The BS-t distribution A RV D following the BS-t distribution with shape $\alpha > 0, \nu > 0$ and scale $\beta > 0$ parameters is denoted by $D \sim \text{BS-}t(\alpha, \beta, \nu)$. In this case, the PDF and CDF of D are

$$\begin{aligned} f_D(d) &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\,\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \left[1 + \frac{\xi^2(d/\beta)}{\nu\alpha^2}\right]^{-\frac{\nu+1}{2}} \frac{[d/\beta]^{-1/2} + [d/\beta]^{-3/2}}{2\alpha\beta} \quad \text{and} \\ F_D(d) &= \frac{1}{2}\Big[1 + I_{\frac{\xi(d/\beta)}{\xi(d/\beta)+\nu}}(1/2, \nu/2)\Big], \quad d > 0, \end{aligned}$$

where $I_a(b, c)$ is the incomplete beta function ratio. The corresponding QF is again $d(q) = F_D^{-1}(q) = \beta[\alpha z(q)/2 + \sqrt{(\alpha z(q)/2)^2 + 1}]^2$, for $0 < q < 1$, but now $z(q)$ is the QF of the Student-t distribution with ν degrees of freedom. Note that β is once again the median or 50th percentile of the distribution. The mean and variance of D are $\text{E}[D] = \beta[1 + A\alpha^2/2]$ and $\text{Var}[D] =$

$\beta^2\alpha^2[A + 5B\alpha^2/4]$, where $A = \nu/[\nu - 2]$, for $\nu > 2$, and $B = 3\nu^2/[(\nu - 2)(\nu - 4)]$, for $\nu > 4$. Now, BS RVs (D) and Student-t (Z) are related by $D = \beta[\alpha Z/2 + (\alpha Z/2)^2 + 1]^2 \sim$ BS-$t(\alpha, \beta; \nu)$ and $Z = [1/\alpha]\xi(D/\beta) \sim t(\nu)$. In this case, $W = Z^2$ follows a Fisher distribution with one degree of freedom in the numerator and ν degrees of freedom in the denominator, which also is useful for goodness of fit purposes. Some of its properties are: $c\,D \sim$ BS-$t(\alpha, c\,\beta, \nu)$, with $c > 0$, and $1/D \sim$ BS-$t(\alpha, 1/\beta, \nu)$.

The gamma distribution A RV D following the gamma distribution with shape $\alpha > 0$ and scale $\beta > 0$ parameters is denoted by $D \sim \text{Gamma}(\alpha, \beta)$. In this case, the PDF and CDF of D are

$$f_D(d) = \frac{d^{1/\alpha^2 - 1}\exp(-d/\alpha^2\beta)}{[\alpha^2\beta]^{1/\alpha^2}\Gamma(1/\alpha^2)} \quad \text{and} \quad F_D(d) = \frac{\gamma(1/\alpha^2, d/\alpha^2\beta)}{\Gamma(1/\alpha^2)}, \quad d > 0,$$

where $\Gamma(\cdot)$ and $\gamma(\cdot, \cdot)$ denote the usual and incomplete gamma functions, respectively. The corresponding QF given by $d(q) = F_D^{-1}(q)$, for $0 < q < 1$, must be obtained by solving this equation with an iterative numerical method. The mean and variance of D are $\text{E}[D] = \alpha\beta$ and $\text{Var}[T] = \alpha^2\,\beta^2$, respectively. The gamma distribution also shares the property $c\,D \sim \text{Gamma}(\alpha, c\,\beta)$, with $c > 0$.

The inverse Gaussian distribution A RV D following the IG distribution with mean $\lambda > 0$ and scale $\beta > 0$ parameters is denoted by $D \sim \text{IG}(\lambda, \beta)$. In this case, the PDF and CDF of D are

$$\begin{aligned} f_D(d) &= \sqrt{\frac{\beta}{2\pi d^3}}\exp\left(-\frac{\beta\,[d - \lambda]^2}{2d\lambda^2}\right) \quad \text{and} \\ F_D(d) &= \Phi\left(\sqrt{\tfrac{\beta}{\lambda}}\,\xi\left(\tfrac{d}{\lambda}\right)\right) + \Phi\left(\sqrt{\tfrac{\beta}{\lambda}}\left[\sqrt{\tfrac{d}{\lambda}} + \sqrt{\tfrac{\lambda}{d}}\right]\right)\exp\left(\tfrac{2\beta}{\lambda}\right), \quad d > 0, \end{aligned}$$

and once again the corresponding QF given by $d(q) = F_D^{-1}(q)$, for $0 < q < 1$, must be obtained by solving this equation with an iterative numerical method. The mean and variance of D are $\text{E}[D] = \lambda$ and $\text{Var}[D] = \lambda^3/\beta$, respectively. The IG distribution also shares the scale property, that is, $c\,D \sim \text{IG}(c\,\lambda, c\,\beta)$, with $c > 0$.

The lognormal distribution If $Y = \log(D)$ has a normal distribution with mean μ and variance α^2, that is, $Y = \log(D) \sim \text{N}(\mu, \alpha^2)$, then the RV D follows the LN distribution with shape $\alpha > 0$ and scale $\beta = \exp(\mu) > 0$ parameters. The notation $D \sim \text{LN}(\alpha, \beta)$ is used in this case. Thus, the PDF and CDF of D are

$$f_D(d) = \frac{1}{d\,\alpha\,\sqrt{2\pi}}\exp\left(-\frac{[\log(d) - \log(\beta)]^2}{2\alpha^2}\right) \text{ and } F_D(d) = \Phi\left(\frac{\log(d) - \log(\beta)}{\alpha}\right),$$

for $d > 0$. The corresponding QF is $d(q) = F_D^{-1}(q) = \beta\exp(z(q)\alpha)$, for $0 < q < 1$, where $z(q)$ is the standard normal QF. The mean and variance of D are $\text{E}[D] = \beta\exp(\alpha^2/2)$ and $\text{Var}[D] = \beta^2[\exp(2\alpha^2) - \exp(\alpha^2)]$, respectively.

The Weibull distribution A RV D following the Weibull distribution with shape $\alpha > 0$ and scale $\beta > 0$ parameters is denoted by $D \sim \text{Wei}(\alpha, \beta)$. In this case, the PDF and CDF of D are

$$f_D(d) = \frac{\alpha\, d^{\alpha-1}}{\beta^\alpha} \exp\left(-\left[\frac{d}{\beta}\right]^\alpha\right) \quad \text{and} \quad F_D(d) = 1 - \exp\left(-\left[\frac{d}{\beta}\right]^\alpha\right), \quad d > 0.$$

The QF of D is $d(q) = \beta[-\log(1-q)]^{1/\alpha}$, for $0 < q < 1$, and its mean and variance

$$\text{E}[D] = \beta\,\Gamma\left(\frac{\alpha+1}{\alpha}\right) \quad \text{and} \quad \text{Var}[D] = \beta^2\left[\Gamma\left(\frac{\alpha+2}{\alpha}\right) - \left\{\Gamma\left(\frac{\alpha+1}{\alpha}\right)\right\}^2\right].$$

Data analysis, parameter estimation and goodness-of-fit of distributions As mentioned, R is a free software environment for statistical computing and graphics. Using this software (i) exploratory data analysis (EDA) can be conducted for diagnosing the statistical features present in the demand data; (ii) estimation of the parameters of the BS, BS-t, gamma, IG, LN and Weibull distributions can be carried out by the popular maximum likelihood (ML) method, and (iii) goodness-of-fit of a distribution to a demand data set can be performed by Anderson-Darling (AD) and Kolmogorov-Smirnov (KS) tests and probability plots. Next, we describe the R commands of the gbs, ig and basics packages and briefly illustrate their use.

First, the R software must be downloaded from `CRAN.r-project.org` and installed as any other software. Second, this software can be used in a simple interactive form with the R commander by installing the Rcmdr package. Third, the gbs and ig packages must be also installed. Data analyses based on the BS and BS-t distributions can be carried by the gbs package, for the IG distribution with the ig package and for the gamma, LN and Weibull distributions with the basics or gamlss packages. Thus, once these packages are installed, they must be loaded into the R software, for example, by the command `library(gbs)` typing them at the R prompt of the R commander, or at any editor program that the user is considering. Once all these instructions are ready, the data, for example, "component1" say, must be loaded as `data(component1).` The data can also be directly typed by the R commander such as an Excel sheet or imported from text files, from other statistical software or from Excel. Table 3.1 provides examples of some commands that allow us to work with the BS distribution, whereas analogous instructions can be used for the other distributions; for more details about how using the gbs package, see Barros et al. (2009).

1.2.4 Inventory management models

Once the most suitable demand distribution is chosen from the pool presented above, the appropriate inventory management model must be applied considering if the product (component) is (i) perishable –under a single period–, (ii) non-perishable –under multiple periods–, or (iii) supplied with the JIT method. Thus, depending on the type of product, we obtain the quantity to be replenished that minimizes the PCs, OCs and SCs according to one of the following inventory models:

(M1) Model for non-perishable products or (Q, r): it considers that the quantity needed to optimize the OCs and SCs is based on the EOQ model given by

$$Q = \sqrt{\frac{2\,\lambda\,\text{OC}}{\text{SC}}}, \tag{1.1}$$

Table 1.1: basic functions of the gbs package.

Function	Instruction	Result
PDF	dgbs(1.0, alpha=0.5, beta=1.0)	0.798
CDF	pgbs(1.0, alpha=0.5, beta=1.0)	0.500
QF	qgbs(0.5, alpha=0.5, beta=1.0)	1.000
numbers	rgbs(n=100, alpha=1.0, beta=1.0)	It generates 100 BS(1, 1) random numbers
MLE	mlegbs(x)	It estimates the BS parameters by the ML method using the data x.
EDA	descriptiveSummary(x)	It provides a summary with the most important descriptive statistics.
histogram	histgbs(x, boxPlot=T, pdfLine=T)	It produces a histogram and a boxplot with the estimated BS PDF using the data x.
envelope	envelopegbs(x)	It produces a probability plot with envelope using the data x.
KS test	ksgbs(x, graph=T)	It computes KS p-value and plots of estimated theoretical BS and empirical CDF using data x.

where λ is the demand rate in units of the product per time unit, calculated as the mean (expected value) according to the distribution that adequately fits the data. In the model M1, we must consider also the ROP, which corresponds to the level that an inventory must to have in stock when a purchase order is placed, calculated as $r' = l\,\lambda$, where λ is the mean of the demand and l is the constant LT. However, due to consumption is occurring in a random fashion provoking demand uncertainties, to be protected against such an uncertainty, it is necessary to include a SS doing the ROP becomes

$$r = \mu_{D_l} + \text{SS}, \tag{1.2}$$

where $\mu_{D_l} = \text{E}[D_l] = l\,\lambda$ is the mean of the demand during the LT (D_l) and $\text{SS} = k_q\,\sigma_{D_l}$, with k_q being the SF associated with a service level $q \times 100\%$, or amount of standard deviations (SD) of the demanded quantity during the LT given by $\sigma_{D_l} = \sqrt{\text{Var}[D_l]} = \sqrt{l}\,\sigma$. As noted in (1.2), it is necessary to know the demand distribution during the LT to determine the SF; see Keaton (1995). This factor can be established by using some percentile of the demand during the LT. To be protected against any unexpected situation of logistics, the 95th percentile is usually considered, that is, $q = 0.95$. Thus, $k_{0.95}$ must be obtained from the statistical distribution that adequately fits the demand data during LT.

Note that in the model M1 is not considered a shortage cost of the product, because, in case of occurring shortage, it is possible to produce a menu of emergency, so that no unsatisfied demand is generated for the final product (menu); see details about this model and it assumptions in (Hillier and Lieberman, 2005, pp. 956-961). Also, we recall no simultaneous optimization of Q and r is carried out due to the practical nature of our methodology.

(M2) Model for perishable products: it considers the quantity needed to optimize the cost of ordering one unit less (generating a temporary shortage), in contrast to ordering one unit more (generating a temporary overstock), based on the CR in this case given by $\text{CR} = [\text{UC} - \text{PC}]/[\text{UC} + \text{HC}]$, where UC is the unsatisfied demand shortage cost per unit, that includes lost revenue and loss cost of customer goodwill, PC is expressed as a purchasing cost per unit of the product, and HC is the holding cost per unit each day, that includes the SC minus a salvage value of a product unit. The numerator $\text{UC} - \text{PC}$ results in the decrease in profit, due to not ordering a unit that could have been sold during such a period, whereas the denominator $\text{UC} + \text{HC}$ results also in the decrease in profit, but due to ordering a unit that could not be sold during such a period. Thus, the single period model for perishable products allows us to obtain its optimum stored quantity from the optimum service level given by

$$F_D(d^0) = \text{CR}, \tag{1.3}$$

where $F_D(\cdot)$ is the CDF of the demanded quantity and d^0 the optimum quantity of ordered units; see details in (Hillier and Lieberman, 2005, pp. 961-975).

(M3) JIT model: it is the just quantity for production, does not consider storage and is used for specific products requested for completing the daily menu of the food service company. A Kanban type information system can be used in this case, which allows the availability of the product to be harmonically coordinated; see Carter et al. (2000).

1.2.5 Determination of financial indicators

Once an appropriate inventory management model is chosen from M1, M2 or M3, the CM for each of p products (components) of the inventory assortment used in the preparation of a food menu portion must be calculated, based on the incomes obtained during w weeks for the company corresponding to this menu. The quantities of each component used in the preparation of the menu (ingredients) are determined with its respective consumption measuring unit; see Table 3.13 in Appendix for an example about equivalence among these units for the products of the case study that we conduct in Section 1.3.

The prorated demand of the product i in the jth week can be obtained by means of the proportion that each product of the food portion holds weekly in the menu calculated according to

$$\text{PD}_{i,j} = \text{DQ}_{i,j}/\text{DQ}_j, \quad i = 1, \ldots, p, \quad j = 1, \ldots, w, \tag{1.4}$$

where $\text{DQ}_{i,j}$ is the demanded quantity of the product i in the jth week and DQ_j is the demanded quantity for all the products during that week. The income of the company for all the portions of the food menu sold during the jth week is

$$\text{I}_j = \text{N}_j\, \text{S}_j, \quad j = 1, \ldots, w, \tag{1.5}$$

where N_j is the number of menus sold and S_j is the price of the food menu portion, both of them in the jth week. Thus, the prorated income due to the product i during the jth week is obtained as

$$\text{PI}_{i,j} = \text{I}_j\, \text{PD}_{i,j}, \quad i = 1, \ldots, p, \quad j = 1, \ldots, w, \tag{1.6}$$

where $\text{PD}_{i,j}$ and I_j are defined in (1.4) and (1.5), respectively. The PC for the product i in the jth week is

$$\text{PC}_{i,j} = \text{NC}_{i,j}\, \text{PQ}_{i,j}, \quad i = 1, \ldots, p, \quad j = 1, \ldots, w, \tag{1.7}$$

where $\text{NC}_{i,j}$ and $\text{PQ}_{i,j}$ are the unit net cost and the purchased quantity of the product i during the jth week, respectively. Note that, for the optimized system with the inventory model for non-perishable products, $\text{PQ}_{i,j}$ must be estimated from Q given in (3.7), whereas that, in the case of perishable products, $\text{PQ}_{i,j}$ must be estimated from $d^0 - L_j$ given in (2.5), with L_j being the stock level at the beginning of the jth week. For the non-optimized system, this value can be empirically calculated. Once financial indicators $\mathrm{PI}_{i,j}$ and PC_j defined in (1.6) and (1.7) are obtained, the variable contribution margin (VCM) of the product i during the jth week must be computed as

$$\mathrm{VCM}_{i,j} = \mathrm{PI}_{i,j} - \mathrm{PC}_{i,j}, \quad i = 1, \ldots, p, \quad j = 1, \ldots, w. \tag{1.8}$$

The OC for the product i during the jth week can be obtained as

$$\mathrm{OC}_{i,j} = \mathrm{OC}_i/52, \quad i = 1, \ldots, p, \quad j = 1, \ldots, w, \tag{1.9}$$

where OC_i is the annual OC of the product i given by $\mathrm{OC}_i = \sum_{h=1}^{3} \mathrm{OC}_i^h \, \mathrm{OQ}_i$, with OQ_i being the annual order quantity and OC_i^h the cost of type h given in Table 3.3, both for the product i. Note that, for the optimized system with the inventory model for non-perishable products, OQ_i must be estimated from Q given in (1.1) using the expression λ/Q for each product (with λ being expressed as a demand rate per year), whereas that in the case of perishable products $\mathrm{OQ}_i = 52$, for all $i = 1, \ldots, p$. For the non-optimized system, this value can be empirically calculated.

Table 1.2: costs involved in generating a purchase order (OC^h).

Cost	Description
OC^1	Administrative costs associated with the order movements (input and general service costs with respect to order generation).
OC^2	Inspection and receiving costs (social security contributions and warehouseman wages) of movements associated with an order.
OC^3	Transportation costs related solely to order generation.

The SC for the product i during the jth week is

$$\mathrm{SC}_{i,j} = [\mathrm{SC}_i/52] \, \mathrm{SQ}_{i,j}, \quad i = 1, \ldots, p, \quad j = 1, \ldots, w, \tag{1.10}$$

where SC_i is the annual SC of the product i given by $\mathrm{SC}_i = \sum_{k=1}^{5} \mathrm{SC}_i^k/\mathrm{SQ}_i$, with SC_i^k being the annual SC of type k defined in Table 1.3 and $\mathrm{SQ}_i = \sum_{j=1}^{52} \mathrm{SQ}_{i,j}$ the annual stored quantity, both for the product i, and $\mathrm{SQ}_{i,j}$ is the stored quantity of the product i in the jth week. Note that, for the optimized system with the inventory management model for non-perishable products, $\mathrm{SQ}_{i,j}$ must be estimated from $\mathrm{SQ} = Q/2 + \mathrm{SS}$, where Q and SS are given in (1.1) and (1.2), respectively, whereas that, in the case of perishable products, $\mathrm{SQ}_{i,j}$ must be estimated from the expected inventory level by single period. For the non-optimized system, this value can be empirically calculated.

We consider CMs as absorbable by the sales with respect to indirect costs, which are subtracted from the VCM given in (1.8) to obtain the total CM of the product i during the jth week as

$$\mathrm{CM}_{i,j} = \mathrm{VCM}_{i,j} - [\mathrm{OC}_{i,j} + \mathrm{SC}_{i,j}], \quad i = 1, \ldots, p, \quad j = 1, \ldots, w, \tag{1.11}$$

where $\mathrm{VCM}_{i,j}$, $\mathrm{OC}_{i,j}$ and $\mathrm{SC}_{i,j}$ are given in (1.8), (1.9) and (1.10), respectively. Thus, we collect a series of CMs for p products (one for each of them). Hence, the CM of all the products of the

Table 1.3: annual costs involved in the storage of a product (SC^k).

Cost	Description
SC^1	Annual cost of amortization of buildings and networks for air conditioning, handling equipment, information processing, receiving, storage media and weighing, among others.
SC^2	Annual cost of damage, losses, obsolescence and product losses incurred in the storage period.
SC^3	Annual cost of cleaning materials and storehouse, containers, packaging, and printed matter.
SC^4	Annual cost of energy spent on the storehouse, including battery charging necessary for handling, data processing equipment and lighting.
SC_5	Annual cost of rental of equipment and facilities, during insurance, storage and communications, and taxes.

inventory assortment during the jth week is $\text{CM}_j = \sum_{i=1}^{p} \text{CM}_{i,j}$, for $j = 1 \ldots, w$, where $\text{CM}_{i,j}$ is given in (1.11). Therefore, the total CM of the inventory system is

$$\text{CM} = \sum_{j=1}^{w} \text{CM}_j. \tag{1.12}$$

Note that the objective function to be maximized is the sum of $\text{CM}_{i,j}$ for the product i in the jth week, during all the period of study totalizing w weeks, for the menu composed by p products with independent demand. Here, the margins $\text{VCM}_{i,j}$ and costs OC_j and $\text{SC}_{i,j}$ depend on the inventory model of the product i. This function is expressed as

$$\sum_{i=1}^{p}\sum_{j=1}^{w} \text{CM}_{i,j} = \sum_{i=1}^{p}\sum_{j=1}^{w} [\text{VCM}_{i,j} - \text{OC}_{i,j} - \text{SC}_{i,j}].$$

Due to that our approach to calculating (i) CMs from the differential revenues and (ii) costs from the movements in and out of the inventory assortment is based on independent components (products) and not from the menu, the absorbable costs for ordering and storing are also calculated using the same criteria of independence and considering the spread of demand from the proportions of components used in the menu. This approach turns out to be more streamlined, because it does not consider the correlations that could exist between components of the menu, which can be a source for a future work.

1.2.6 Summary of the methodology

Algorithm 1 summarizes our methodology in six main steps divided in 13 sub-steps based on the aspects detailed above, from the collection of data until the establishment of the CMs to evaluate the optimized system in relation to the current (non-optimized) system. We recall this algorithm considers the demand for independent components, but once all the components are considered, the total contribution of the components used in the service are maximized.

Algorithm 1 Main methodological steps

1: Collect demand data for the product i in each day of the w weeks ($i = 1, \ldots, p$).
2: For the statistical analysis:
   2.1 Carry out a correlation study for data collected in Step 1 to detect possible seasonality, trend or dependence. If neither autocorrelation nor correlation between components are detected, then an EDA for independent data must be conducted. Otherwise, these seasonality and/or trend must be removed using suitable techniques.
   2.2 Propose distributions for the demand data analyzed in Step 2.1 based on the EDA.
   2.3 Estimate the parameters of the distributions proposed in Step 2.2.
   2.4 Apply goodness-of-fit tests establishing the most adequate distribution.
3: For the inventory analysis:
   3.1 Select the suitable inventory model depending on the type of product i.
   3.2 Find the optimal inventory elements (Q, r, d^0) based on distributions established in Step 2.4.
4: For the financial analysis:
   4.1 Compute the VCM for the product i in the jth week of the optimal policy obtained in Step 3.2.
   4.2 Determine the corresponding OC for the product i in the jth week.
   4.3 Calculate the SC for the product i in the jth week.
   4.4 Obtain the CM for the product i in the jth week.
5: Repeat steps 1 to 4 until completing p products.
6: Establish the optimized total CM and compare it with the non-optimized total CM.

1.3 Case study

In this section, because there is a need to conduct case studies focusing on their applicability in firms to reduce the gap between theory and practice, we apply the methodology summarized in Algorithm 1 to an anonymous Chilean food company, which serves the staff of a hospital in the city of Valparaiso. This case study enables researchers to increase their practical knowledge given that aspects involving understanding about environment's complexity and the managerial efforts made by firms become evident.

This study was leaded by Fernando Rojas and Victor Leiva in the University of Valparaiso-Chile (`www.uv.cl`) by means of the project grant DIUV 14/2009, during $w = 27$ weeks covering the period since 20-Nov-2011 to 26-May-2012 (189 days). Details of $p = 89$ products of the inventory assortment considered in this study are provided in Table 3.13 with their respective equivalence units.

We recall that the unsatisfied demand shortage costs for non-perishable products are unavailable to be incorporated in the analysis, because there is no CMs or penalties that are imposed for a product with unsatisfied demand. If a product is missing, it is replaced by similar another. In addition, due to the practical character of this study, for non-perishable products, we set a service level based on a SF instead of simultaneously optimizing Q and r.

As mentioned, the data have been collected during the period indicated above following the record system. Note that, in the type of data that we analyze (food services for hospitals), seasonality or trend factors usually are not present; see Step 2 of Algorithm 1. We have also explored the correlation between some products and only a small correlation but marginally not significant is detected, so that we discard this aspect. In any case, some comments in this line are provided in the conclusions of this study. Moreover, demand data are usually observed over time. Then, one must check whether these data have a dependence in the time or not. An autocorrelation graphical analysis detected that the corresponding autocorrelations are very small, so that dependence in the time can be discarded too. This graphical analysis can be corroborated by the Durbin-Watson test and its bootstrapped p-value to examine independence in these data.

Second, we carry out a statistical analysis of these data from the EDA until the selection of the most appropriate distribution for the demand data of each product under study. Tables 3.8-3.9 display a summary of the statistical results for 89 products of the inventory assortment of the Chilean food service. This summary indicates, among other aspects, the statistical distribution that fits the demand data best for each product.

Third, once we have selected the most appropriate distribution to model the demand data, we then use the adequate inventory management model to determine the optimum level in stock to place an order of products, and the optimum quantity to order for minimizing the total cost of inventory. Tables 3.8-3.9 (see Appendix) also show the optimal quantity of replenishment and the ROP obtained by applying the appropriate inventory management model.

In Tables 3.8-3.9, note that "P" and "EOQ" are the "inventory model" for the corresponding perishable and non-perishable product, respectively; "statistical distribution" corresponds to the fitted demand distribution for the indicated product according to the ID detailed in Table 3.13 (see Appendix); λ and σ are the estimated demand mean rate and SD given in (1.1) and (1.2); $k_{0.95}$ is the SF for a service level of 95% given below (1.2); "SS" is the safety stock given in (1.2); Q is the EOQ given in (1.1); CR and d^0 are given in (1.3); r is the ROP given in (1.2); and "JIT" is considered when this method is used for such a product. Note also that the symbol "-" is used when the corresponding value must not be calculated; "average" indicates that any distribution can be fitted for such a product and then the normal distribution is used. From Tables 3.8-3.9, we describe 38 of the 89 products (components) from the inventory assortment with the EOQ model given in (1.1), 47 with the perishable model given in (1.3), and 4 with the JIT method, indicating that the total inventory is made up of mostly perishable type products. BS distributions were adequate for several of the demand data sets in those products allowing a distribution to be fitted (not JIT).

Fourth, once we have calculated the elements of the inventory models by using equations (1.1) and (1.3), following the financial approach detailed above, we compute the differences between direct and indirect costs (unit and annual), with weekly and annual ordering, and obtain the differences between the CMs with respect to the entire product inventory assortment of non-optimized and optimized systems, by using equations (1.4) to (1.12). Table 1.4 shows the annual and weekly OCs in both systems, where OCs for the optimized system increase 54.64%.

Table 1.4: annual and weekly OCs (in US$) for the indicated system.

OC^h	Non-optimized system	Optimized system
OC^1	142.40	220.26
OC2	1602.03	2478.01
OC^3	2848.06	4405.35
Total	4592.50	7103.62
order/week	1.14	1.76
OC/order	77.44	77.44

Table 1.5 shows the annual SCs of the non-optimized and optimized systems, which are diminished in 84.05%, translating our proposal into a significant saving due to the improvement obtained by using the inventory management models.

Table 1.5: annual SCs (in US$) for the indicated system.

	Non-optimized system			Optimized system		
SC^k	SC(a)[1]	SC(w)[2]	SC(au)[3]	SC(a)	SC(w)	SC(au)
SC^1	2754.24	52.97	0.011	575.85	11.07	0.0030
SC^2	11546.61	222.05	0.045	1207.06	23.21	0.0063
SC^3	1784.32	34.31	0.007	373.06	7.17	0.0019
SC^4	7627.12	146.68	0.030	1594.65	30.67	0.0019
SC^5	635.59	12.22	0.002	132.89	2.56	0.0007
Total	24347.88	468.23	0.095	3883.51	74.68	0.0202

[1]SC(a) is the annual SC in US$. In non-optimized and optimized systems, 919962.6 and 192342.9 unit/year are stored, respectively; [2]SC(w) is the weekly SC in US$; [3]SC(au) is the annual SC in US$.

Tables 3.11-3.12 (see Appendix) show the differential of VCM, OC, SC and CM values for all the products in an descendent order, obtained by subtracting the results from the non-optimized and optimized systems, for each of these financial indicators. A positive value of the differential indicates a saving is detected for the indicated product, using the optimized system. A negative value indicates the attained optimization is unfavorable in that case for the indicated product. In Tables 3.11-3.12, the set of critical products, which account about 80% of the optimized values obtained in these financial indicators have been delimited by a line. This is established as a cumulative percentage (CP) of optimized values regarding the total optimization attained in the differential profit or saving of the financial indicator, using the classification ABC; see details in Ramanathan (2006).

1.4 Illustration

In this section, we illustrate the optimized analysis of one of the 89 products of the inventory assortment of the case study presented in Section 1.3. We select this product due to its statistical features, so that a practitioner can understand in a better way how the analysis is produced for a component, which can be replicated for other components. This analysis is divided in three parts following Steps 2, 3 and 4 of Algorithm 1.

1.4.1 Statistical analysis

The data correspond to the demanded amount (D) of the ground beef product (in kg) with ID = P42, which was collected during the period under study. Table 1.6 displays a descriptive summary of the demand data that includes the sample median (50th percentile), mean ($\bar{d}$), SD, coefficients of variation (CV), skewness or asymmetry (CS) and kurtosis (CK), and sample size (n), among other statistics. This summary is obtained by the command `descriptiveSummary()` of the gbs package. From Table 1.6, we note that the CS and CK for P42 data show a distribution with positive skewness and moderate kurtosis.

Table 1.6: descriptive measures for P42 data (in kg).

n	Min	Med	$\bar{d}$	SD	CV	CS	CK	Range	Max
68	1.00	17.00	20.37	15.34	75.33%	1.08	3.43	64.00	65.00

Figure 1.1 shows the histogram, boxplot and graph of the empirical CDF (ECDF) for P42 data. These graphs are built with the command `histgbs()` of the gbs package and `boxplot()` and `ecdf()` of the base R package. Note that: (i) the histogram shows a PDF with positive skewness and moderately heavy tails; see also Table 1.6; and (ii) the boxplot displays some outliers. Based on the EDA results, BS distributions seem to be good options for modeling P42 data, because they can accommodate their outliers and degrees of variability, skewness and kurtosis.

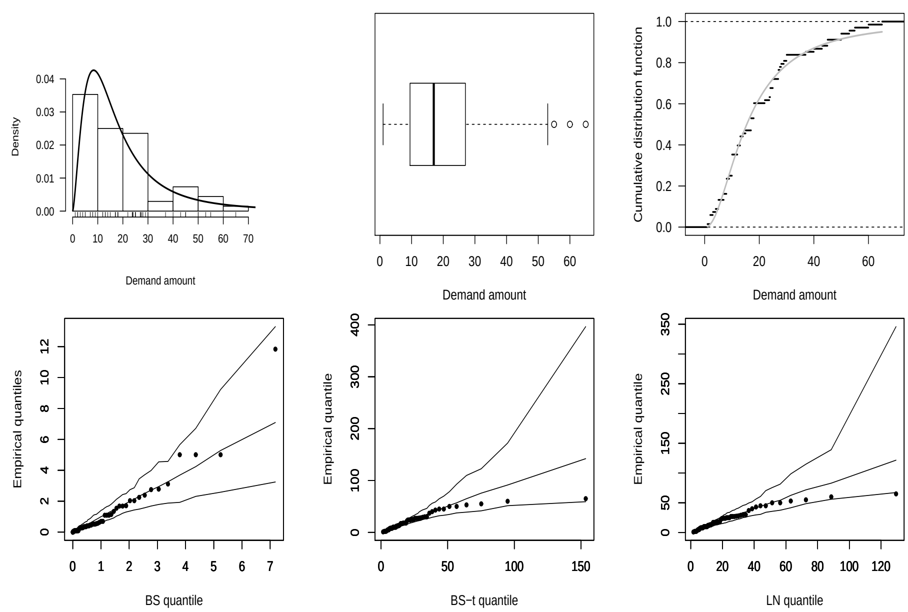


Figure 1.1: [first panel] histogram with estimated BS-t PDF (left), boxplot (center) and ECDF with estimated BS-t CDF (right) and [second panel] plots of probability with envelopes for the indicated distribution using P42 data.

BS, BS-t, gamma, IG, LN and Weibull parameters can be estimated using the ML method; see Barros et al. (2009). For this purpose, commands `mlegbs()` and `gamlss()` of the gbs and gamlss packages, respectively, can be used. The goodness of fit of the model to P42 data can be checked using the AD and KS tests, which compare the ECDF and the theoretical CDF assumed for the data (within BS, BS-t, gamma, IG, LN and Weibull models). The command used for obtaining these results is `ksgbs()` of the gbs package, and its corresponding adaptations to the gamma, IG, LN and Weibull distributions. Table 1.7 provides the p-values of the AD and KS tests for P42 data, from which we note that almost all of these distributions seem to be reasonable models for these data. However, based on the AD test results, which is more powerful than the KS test Barros et al. (2014), only the BS, BS-t and LN models fit the data well at a significance level of 1%.

Table 1.7: p-values of the indicated method and distribution for P42 data.

Method	BS	BS-t	Gamma	GI	LN	Weibull
AD	0.0143	0.1214	<0.001	<0.001	0.0604	0.0022
KS	0.1668	0.7562	0.1525	0.0017	0.6165	0.2338

The fit of the model to P42 data is visually illustrated in Figure 3.1, from where the ECDF (gray line) and the theoretical BS-t CDF (black dots) are compared on the right, whereas the histogram with the estimated BS-t PDF is plotted on the left. Probability plots with envelopes are shown in Figure 3.1, where "envelopes" are bands constructed by a simulation process facilitating the display setting. In the case of BS distributions, these envelopes are built using the formulas given above; see details in Leiva et al. (2011a) and references therein. The commands used to obtain these graphs are `envelopebs()` and `envelopegbs()` and their corresponding adaptations to the gamma, IG, LN and Weibull distributions. From these graphs, we note the appropriate fitting provided by BS, BS-t and LN distributions proposed for modeling P42 data, but also the gamma and Weibull models (omitted here) are suitable, where all the point are inside of their envelopes, which corroborates the results provided in Table 1.7. However, as can be seen from the boxplot given Figure 3.1(center), there are some outliers that can introduce an adverse effect on the ML estimates of the parameters of the distributions detected as suitable by the goodness-of-fit methods. Nevertheless, as mentioned, only the BS-t distribution has been proven to provide estimates robust to these outliers. Thus, we choose the BS-t distribution as the most suitable within the distributions proposed for describing P42 data.

1.4.2 Inventory analysis

Once we have selected the BS-t distribution as the most suitable one to describe the quantity demanded of P42, we use the perishable product model for single period to determine the optimum quantity to be ordered for minimizing the total cost of inventory. First, we estimate the demand rate from the BS-t distribution as $\widehat{\lambda} = 23.19$ kg/day. Then, with this value, we determine the optimum replenishment quantity as $\widehat{d^0} = 15.45$ kg, by using the formula given in (1.3), whose value must be applied as refueling. We consider a constant LT of $l = 3$ days, which is the same for all the products of the inventory assortment. Thus, at the beginning of each week, the stock level must be checked, and then a quantity of $(15.45 - L_j)$ kg, for $j = 1, \ldots, 27$, of the product must be ordered.

Note that, for the case of non-perishable products, once again we first estimate the demand rate and, then, with this estimate, we calculate the optimum Q and r using the formulas given in (1.1) and (1.2), respectively. Thus, when the stock level is in r units, we generate an order of Q units of this type of product.

1.4.3 Financial analysis

Once we have chosen the appropriate inventory model for the P42 product, we determine its CMs according to expression given in (1.11). First, we obtain the VCMs based on (1.8) following the definitions and the sequence of equations given in (1.4)-(1.7). Second, we calculate the corresponding OC and SC using the formulas displayed in (1.9) and (1.10), obtaining $\text{OC}_{42,j} = \text{US\$}0.88$

per order and $SC_{42,j} = US\$0.085$ per stored unit of the P42 product. With this, we obtain $CM_{42} = -US\$5242.72$ (optimized value) in comparison to $-US\$5613.85$ (non-optimized value), reaching a reduction of 6.61% for this product. Table 3.10 provides the weekly values of SQs, VCMs, OCs, SCs and CMs for the non-optimized and optimized systems in the P42 product.

Chapter 2

Exploring the potential use of the Birnbaum-Saunders distribution in inventory management

2.1 Introduction

Inventory management permeates decision-making in countless firms. The topic has been extensively studied in academic and corporate spheres, e.g., Braglia et al. (2004b) and Cai et al. (2014). The key questions which the inventory management seeks to answer –usually influenced by a variety of circumstances– are: when to order, determining an economic order quantity (EOQ) or lot size, and how much safety stock (SS) to keep, establishing a reorder point (ROP); see Namit and Chen (1999) and Porras and Dekker (2008).

According to Wanke (2008a), inventory management involves a set of decisions whose objective is to match existing demand with the supply of products and materials over space and time. This objective allows us to achieve specified costs and service levels, considering product, operation and demand characteristics. It is known that the inventory total cost (TC) is a function of ordering, holding and shortage costs; see Hillier and Lieberman (2005).

The importance attached by firms to inventory management can be attributed to the following. First and foremost, to the need to assure that products, given the competitive pressure exercised by markets, are always supplied to customers at the least possible cost; see Eaves (2002). Second, some other factors contribute to a high concern with inventory management, such as product diversity or behavior; see Huiskonen (2001). High opportunity costs also contribute to this concern, thus affecting the financial indicators on which assessments of firm performance are based; see Wanke (2008a).

The inventory management models are frequently classified in two types: pull and push. On the one hand, according to Ballou and Burnetas (2003), pull-type planning models range from those that set inventory levels based on the EOQ to those fixed in proportion to forecasted demand. The EOQ model is the simplest and most fundamental of all inventory models because it describes important trade-offs between fixed ordering and holding costs; see Nahmias (2001). Despite its shortcomings, the basic EOQ model is the cornerstone of several software packages for inventory control; see H. and S. (1993). Interested readers can refer to Yan and Wang (2013) and Min et al.

(2014) for more details about the EOQ model. Today EOQ is used in conjunction with ROP in inventory control models to determine cycle and SS under DPUT and LT uncertainty. These models are well described in most logistics and operations textbooks, as are their underlying assumptions; see Nahmias (2001). Models that set inventory levels in proportion to forecasted demand constitute a particular form of the periodic review model, except that replenishment quantities are not based on the EOQ model; see Ballou (2005). On the other hand, push-type planning models take place when inventory decisions are based on the demand or its forecast at multiple downstream stocking locations, similar to the resource planning system of logistics.

According to Silver et al. (1998), Eaves and Kingsman (2004), Syntetos et al. (2005) and Boylan et al. (2008), demand is one of the main factors in inventory management models. In general, demand may be classified from two settings. First, it can be deterministic or random; if random, it follows a statistical distribution (also known as probabilistic model); otherwise, the demand is constant, which implies a degenerate statistical distribution, that is, its variance is equal to zero. Second, demand might be independent or dependent. For more details, see Disney et al. (2003), Porras and Dekker (2008), Wanke (2008a) and Rojas et al. (2015).

Demand uncertainties directly affect the operation of the physical system of logistics. Moreover, to be closer to reality, single or multiple period inventory models must take into account that demand is occurring in a random fashion, which is explained by several factors. Thus the demand per unit of time (DPUT) is taken to be a random variable (RV). Furthermore, during the lead-time (LT), due to this randomness, the corresponding demand (LTD) is also a RV; therefore, the behavior of DPUT and LTD must be described by statistical distributions; see Johnson et al. (1994, 1995). The Gaussian (or normal) distribution is often used for describing the data of these two RVs (DPUT, LTD) involved in inventory models. However, it is well-known that the normal distribution is validly used for RVs that take negative and positive values with a symmetrical behavior. Hence, first, quantities less than zero could be admitted when the modeling is carried out under the normal distribution, which is not possible in real-world situations for DPUT and LTD, because they only admit values greater than zero; see Nahmias (2001). Second, another drawback using the normal model is that DPUT and LTD data often follow asymmetric distributions; see Moors and Strijbosch (1988). Mentzer and Krishnan (1988) studied the non-normality effect on inventory models and found that the normal distribution is appropriate in few practical cases; see also Eppen and Martin (1988). A recent case study with DPUT data of 89 food products supports such non-normality; see Leiva et al. (2016) and Rojas et al. (2015). In any case, the normality assumption must be checked by goodness-of-fit methods; see Barros et al. (2014). Thus, the use of the normal distribution to model DPUT and LTD, and then to determine the ROP and SS, can lead to wrong results, resulting in shortages or excess inventories. Non-normal distributions with positive support that have been used for describing DPUT in inventory management include such models as gamma or Erlang, inverse Gaussian, log-normal, Pearson, Poisson, uniform and Weibull; see Burgin (1975), Tadikamalla (1981), Lau (1989), Wanke (2008c), Cobb et al. (2013) and Pan et al. (2014).

A unimodal, two-parameter probability model with positive support and asymmetry to the right that is receiving considerable attention is the Birnbaum-Saunders (BS) distribution; see Birnbaum and Saunders (1969) and Johnson et al. (1995, pp. 651-663). The BS distribution has good properties and is related to the normal distribution and implemented in the `R` statistical software (`www.r-project.org`) via a package called `gbs`; see R-Team (2015). Although the BS distribution has its genesis from engineering, its applications range across diverse fields as business, industry and management, which have been conducted by an international, transdisciplinary group of researchers; see, e.g., Jin and Kawczak (2003), Podlaski (2008), Bhatti (2010), Lio et al. (2010), Paula et al. (2012), Marchant et al. (2013) and Leiva et al. (2014b, 2014e). In addition, although originally conceived as a count model, the BS distribution includes the duration of the counting period (daily or weekly), which obviates having to collect additional data, among other properties; see Fox et al. (2008). In sum, the BS distribution is a good candidate for describing demand data in inventory models; see Leiva et al. (2016) and Rojas et al. (2015).

Our main objective is to explore the use of the BS distribution in inventory management. Differently from previous studies that exclusively considered the effects of one given distribution on inventory decision-making, we also analyze its adequacy in light of different operating characteristics and costs. Specifically, we assess how the BS, gamma and normal LTD distributions interact with relevant product characteristics and affect the optimal EOQ and SS inventory indicators in terms of the optimization of the TC function. We minimize this function using stochastic programming, a technique where constraints and/or objective function of the problem to be optimized contain RVs that can follow any distribution; see Shapiro et al. (2014) and Thangaraj et al. (2010). We solve the problem of stochastic programming with a search heuristic called differential evolution (DE), which is a global numerical optimization approach based on genetic algorithm concepts; see Storn and Price (1997) and Price et al. (2006). We implement our results in `R` code, which is available upon request from the authors.

This chapter is organized as follows. Section 2.2 reviews general aspects of inventory management models and the statistical distributions used in this study. Section 2.3 explores the effect of different LTD distributions in inventory management, introducing the simulation scenario, formulating the stochastic programming model, discussing the DE algorithm, and providing a numerical study.

2.2 Background

In this section, we discuss general aspects of inventory management models and demand statistical distributions used for the methodology presented in Section 2.3.

2.2.1 Inventory management models

The (Q, r) model is based on the ROP (r) and the EOQ model given by

$$Q = \left(\frac{2\,\lambda\,P}{H}\right)^{\frac{1}{2}}, \tag{2.1}$$

where λ is the DPUT rate in units of the product and P, H are the ordering and holding costs, respectively; see Yan and Wang (2013) and Min et al. (2014). However, as mentioned, DPUT is a RV. Then, λ given in (2.1) must be calculated as the mean (expected value) of the DPUT distribution that adequately fits the data. Specifically, let D_t be a RV corresponding to the DPUT at time t, forming a sequence of independent and identically distributed RVs with mean $\mathrm{E}(D_t) = \lambda$

and variance $V(D_t) = \sigma_D^2$. In addition, let L be a RV corresponding to the LT between the ordering of a product and its delivery (expressed in time units) with mean $E(L) = \mu_L$ and variance $V(L) = \sigma_L^2$. Then, the LTD is given by

$$X = \sum_{t=1}^{L} D_t, \tag{2.2}$$

with probability density function (PDF) $f_X(\cdot)$ and whose expectation and variance are, respectively, defined as

$$\mu = E(X) = E(L)E(D) = \mu_L\,\lambda \quad \text{and} \quad \sigma = \sqrt{V(X)} = (\sigma_L^2\,\lambda^2 + \mu_L\,\sigma_D^2)^{1/2}. \tag{2.3}$$

The ROP can be computed from μ expressed in (2.3). However, to be protected from randomness of the LTD, it is necessary to include a SS, which allows the ROP to become

$$r = \mu + k\,\sigma, \tag{2.4}$$

where μ, σ are defined in (2.3), k is the safety factor (SF) or number of standard deviations (SDs) σ of the LTD and $k\sigma$ is the safety stock. Note that although the (Q, r) model is based on (2.1) and (3.1), it is possible to see that r is obtained by k. Thus, we refer to the (Q, r) model as (Q, k) thereafter.

The expected TC of the inventory is given by

$$C(Q, k) = H\left(\frac{Q}{2} + k\,\sigma\right) + \frac{\lambda}{Q}\left(P + S\int_{\mu+k\,\sigma}^{\infty} f_X(u)\,\mathrm{d}u\right), \tag{2.5}$$

where Q (in units of the product) is given in (2.1) and μ, k, σ in (3.1); H is the holding cost (in \$ per \$ per unit of time); P the ordering cost (in \$ per each replenishment order placed); S the shortage cost (in \$ incurred whenever a stock-out occurs); and $f_X(\cdot)$ is the PDF of the LTD given in (2.2). In order to minimize the expected TC defined in (2.5), we optimize the indicator Q given in (2.1) altogether with k given in (3.7).

2.2.2 Demand distributions

Notice that it is necessary to specify the LTD distribution to determine the SS given in (3.1), which allows the SF to be established; see Porras and Dekker (2008). In order to facilitate the calculation of the SF, the LTD has been traditionally modeled with the normal distribution; see Peterson and Silver (1979). Thus, the SF k for a specific service level can be obtained from a percentile of the standard normal distribution, denoted by N(0, 1). However, as mentioned, various studies criticize the normality assumption. Therefore, the use of the normal distribution to determine ROP given in (3.1) , is questionable, leading to possible stock shortage or excess.

Silver (1981) pointed out that in most models leading to inventory management decisions some assumptions are made, often in an implicit way. The effects of these assumptions on costs and service levels should be taken into account. The most common ones are (i) to assume a demand distribution (for example, normal) and (ii) to suppose that the distribution parameters are known (for example, the mean and SD) or estimated from the demand data. Lau (1989) presented a model for computing EOQs and SSs given in (2.1) and (3.1), respectively, using the first four moments, that is, mean, variance, third moment reflecting skewness and fourth moment reflecting kurtosis of any given LTD distribution. Lau (1989) also pointed out the risk of misleading decisions regarding ROP and customer service level when one considers a normally distributed LTD. The 95th percentile of the distribution is often used to set service levels. Next, we present some mathematical features for the three distributions to be considered in this study for the LDT, that is, the BS, gamma and normal models.

The normal distribution A RV X following a normal distribution with mean $\mathrm{E}(X) = \mu$ and variance $\mathrm{Var}(X) = \sigma^2 > 0$ is denoted by $X \sim \mathrm{N}(\mu, \sigma^2)$, where "$\sim$" means "distributed as". In this case, PDF, cumulative distribution function (CDF) and (QF) quantile function of X are, respectively,

$$
\begin{aligned}
f_X(x) &= \frac{1}{(2\pi)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right), \quad x \in \mathbb{R},\\
F_X(x) &= \Phi\left(\frac{x-\mu}{\sigma}\right), \quad x \in \mathbb{R},\\
F^{-1}(q) &= z(q)\,\sigma + \mu, \quad 0 < q < 1,
\end{aligned}
$$

where $\Phi(z) = \int_{-\infty}^{z} \phi(u)\mathrm{d}u$ and $z(q) = \Phi^{-1}(q)$, for $0 < q < 1$, with

$$
\phi(z) = \frac{1}{(2\pi)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}z^2\right), \quad z \in \mathbb{R},
$$

and $\Phi^{-1}(\cdot)$ being the N(0, 1) inverse CDF or QF. In addition, the coefficients of variation (CV), skewness or asymmetry (CS) and kurtosis (CK) of $X \sim \mathrm{N}(\mu, \sigma^2)$ are, respectively,

$$
\mathrm{CV}(X) = \frac{\sigma}{\mu}, \quad \mathrm{CS}(X) = 0, \quad \mathrm{CK}(X) = 3.
$$

The BS distribution A RV X following a BS distribution with shape $\alpha > 0$ and scale $\beta > 0$ parameters is denoted by $X \sim \mathrm{BS}(\alpha, \beta)$. In this case, the PDF, CDF and QF of X are, respectively,

$$
\begin{aligned}
f_X(x) &= \frac{1}{(2\pi)^{\frac{1}{2}}} \exp\left(-\frac{1}{2\alpha^2}\xi^2\left(x/\beta\right)\right) \frac{(x+\beta)}{2\alpha\,\beta^{\frac{1}{2}}\,x^{\frac{3}{2}}}, \quad x > 0,\\
F_X(x) &= \Phi\left(\frac{1}{\alpha}\xi\left(x/\beta\right)\right), \quad x > 0,\\
F^{-1}(q) &= \beta\left(\alpha\, z(q)/2 + \left((\alpha\, z(q)/2)^2 + 1\right)^{\frac{1}{2}}\right)^2, \quad 0 < q < 1,
\end{aligned}
$$

where $\xi(y) = y^{\frac{1}{2}} - y^{-\frac{1}{2}} = 2\sinh(\log(y^{\frac{1}{2}}))$, for $y > 0$, $\Phi(\cdot)$ is the N(0, 1) CDF, $z(q)$ is the N(0, 1) QF, and $F^{-1}(\cdot)$ is the inverse CDF of X. Note that $x(0.5) = \beta$, that is, β is also the median or 50th percentile of the distribution. The mean, variance, CV, CS and CK of $X \sim \text{BS}(\alpha, \beta)$ are, respectively,

$$\begin{aligned}
\text{E}(X) &= \beta\left(1+\frac{\alpha^2}{2}\right), \quad \text{Var}(X) = \beta^2\alpha^2\left(1+\frac{5\alpha^2}{4}\right),\\
\text{CV}(X) &= \frac{\alpha\beta\,(5\alpha^2+4)^{\frac{1}{2}}}{\beta(2+\alpha^2)+2\gamma}, \quad \text{CS}(X) = \frac{4\alpha(11\alpha^2+6)}{(5\alpha^2+4)^{\frac{3}{2}}},\\
\text{CK}(X) &= 3+\frac{6\alpha^2(40+93\alpha^2)}{(4+5\alpha^2)^2}.
\end{aligned}$$

In addition, the RVs $X \sim \text{BS}(\alpha, \beta)$ and $Z \sim \text{N}(0, 1)$ are related by

$$X = \beta\left(\alpha\, Z/2 + \left((\alpha\, Z/2)^2+1\right)^{\frac{1}{2}}\right)^2 \quad \text{and} \quad Z = \frac{1}{\alpha}\xi\left(X/\beta\right).$$

Also, note that $W = Z^2$ follows a chi-squared distribution with one degree of freedom. The BS distribution holds the scale and reciprocal properties, that is, (i) $c\,X \sim \text{BS}(\alpha, c\,\beta)$, with $c > 0$, and (ii) $1/X \sim \text{BS}\,(\alpha, 1/\beta)$, respectively.

The gamma distribution A RV X following a gamma distribution with shape $\alpha > 0$ and scale $\beta > 0$ parameters is denoted by $X \sim \text{Gamma}(\alpha, \beta)$. In this case, the PDF and CDF of X are, respectively,

$$\begin{aligned}
f_X(x) &= \frac{\beta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}\exp(-\beta x), \quad x > 0,\\
F_X(x) &= \frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)}, \quad x > 0,
\end{aligned}$$

where $\Gamma(\cdot)$ and $\gamma(\cdot)$ stand for the usual and incomplete gamma functions, respectively. The corresponding QF given by $F^{-1}(q)$, for $0 < q < 1$, must be obtained by solving this equation with an iterative numerical method. The mean, variance, CV, CS and CK of X are

$$\text{E}(X) = \frac{\alpha}{\beta}, \quad \text{Var}(X) = \frac{\alpha}{\beta^2}, \quad \text{CV}(X) = \frac{1}{\alpha^{\frac{1}{2}}}, \quad \text{CS}(X) = \frac{2}{\alpha^{\frac{1}{2}}}, \quad \text{CK}(X) = \frac{6}{\alpha}.$$

The gamma distribution also shares the scale property, that is, $c\,X \sim \text{Gamma}(\alpha, c\,\beta)$, with $c > 0$.

2.3 Assessing the impact of different distributions

In this section, we introduce our simulation scenario and formulate the stochastic programming used to optimize the expected TC associated with the (Q, k) model. Then, we discuss the DE algorithm, which allows us to solve the problem of stochastic programming, and provide a numerical study performed with the `R` software. We evaluate how different LTD distributions (BS, gamma and normal) interact with different inventory indicators (demand, cost and LT) and their underlying EOQs and SSs. We assess under what circumstances a distributional assumption is preferable to other one in terms of the expected TC.

2.3.1 Scenario of the simulation study

Assume BS, gamma and normal distributions for the LTD. Then, fix values for the parameters of these distributions by considering values for means and SDs of DPUT and LT generated from uniform distributions. Now, generate holding, ordering and shortage costs also from uniform distributions. This allows us to establish the expected TC to be minimized. Ten thousand (10000) different simulated scenarios of means and SDs for DPUT and LT, as well as holding, ordering and shortage costs are generated using an `R` package called `stats`. The values of these uniformly distributed inventory indicators used to build the scenarios are presented in Table 3.3. They were chosen based on values proposed in selected papers focused on managerial and industrial applications compiled by Wanke (2008a); see Table 2.1 in this reference for more details about values that are frequently used to generate simulation scenarios in inventory management problems. A discussion about this can be found in Wanke (2012).

Table 2.1: range of the mentioned uniformly distributed indicator for simulations.

Indicator	Minimum	Maximum
DPUT mean (units/day)	80	120
DPUT SD (units/day)	3	30
LT mean (in days)	1	5
LT SD (in days)	0.50	2.00
Holding cost ($/unit/day)	0.00	0.68
Ordering cost ($/order)	17	60
Shortage cost ($/shortage)	0	100

2.3.2 Stochastic programming

Once the values for the inventory indicators are defined and the distributional assumptions (BS, gamma, and normal) for the LTD established, stochastic programming is performed on the expected TC function given in (2.5); see Namit and Chen (1999). According to the values provided in Table 3.3, we assume values for means and SDs of the DPUT and LT, as well as for the holding, ordering and shortage costs, to be uniformly distributed in the objective function corresponding to the expected TC given in (2.5). The decision variables of the programming are Q and k given in (2.1) and (3.1), respectively, whereas $\lambda, \sigma_D, \mu_L, \sigma_L, H, P, S$ are given from (2.5). Therefore, our optimization problem of the expected TC can be visualized like a model of stochastic programming formulated as:

$$
\begin{aligned}
\text{Minimize} \quad & Z = C(Q, k) \\
\text{Subject to:} \quad & Q > 0, k > 0, \\
\lambda & \sim \text{U}(80, 120), \\
\sigma_D & \sim \text{U}(3, 30), \\
\mu_L & \sim \text{U}(1, 5), \\
\sigma_L & \sim \text{U}(0.50, 20), \\
H & \sim \text{U}(0.00, 0.68), \\
P & \sim \text{U}(17, 60), \\
S & \sim \text{U}(0, 100).
\end{aligned} \tag{2.6}
$$

The problem of stochastic programming formulated in (2.6) is aimed (i) to identify the most adequate inventory policy for each of the distributional assumptions, and (ii) to minimize the expected TC. The optimized problem can provide useful information for academics and practitioners on how these assumptions interact with product, operation and demand characteristics. To solve the problem of stochastic programming formulated in (2.6), we use the DE algorithm detailed in the next section.

2.3.3 Differential evolution

DE is a member of the family of genetic algorithms, which mimic the process of natural selection in an evolutionary manner; see Holland (1975). A genetic algorithm solves optimization problems with biology-inspired operators of crossover, mutation and selection, generating successive populations of individuals (solutions or generations). Then, the DE algorithm optimizes problems by evolving a population of candidate solutions employing the mentioned operators. The DE algorithm uses floating-point techniques for obtaining the solutions and arithmetic operations in their mutation, in contrast to classic genetic algorithms. In addition, the DE algorithm finds the global optimum of the objective function, which is not required to be either continuous or differentiable; see Thangaraj et al. (2010) and Mullen et al. (2011).

The DE algorithm has also been used to optimize problems that arise in inventory management, such as joint replenishment, replenishment coordination and inventory location-allocation; see Qu et al. (2013, 2015) and Wang et al. (2014). In the present study, the problem of stochastic programming formulated in (2.6) is solved with the DE algorithm, which allows us to find the optimal values of Q and k that minimize the expected TC function $C(\cdot)$ given in (2.5), for 10000 simulated scenarios.

In what follows, we first discuss how the DE algorithm used in our research works; see Storn and Price (1997) and Price et al. (2006) for more details about the algorithm. Then, we sketch the content of an `R` package called `DEoptim`, which implements the DE algorithm and was first published on CRAN in 2005. Since becoming publicly available, it has been used by several authors to solve optimization problems arising in diverse domains. We refer interested readers to Ardia et al. (2011) and Mullen et al. (2011) for a detailed description of the package.

Let N be the number of members $\boldsymbol{y} \in \mathbb{R}^k$ (tuning parameter vector) in the population, where k denotes the dimension of the vector $\boldsymbol{y}$. The DE algorithm needs a starting population, which is obtained by sampling the objective function at multiple randomly chosen initial points (generation or solution 0) for Q and k. Before the population is initialized, both lower and upper bounds for each tuning parameter must be specified. Parameter bounds establish the domain from which the N vectors at the generation 0 are chosen. To establish the generation 0, N guesses for the optimal value of $\boldsymbol{y}$ must be provided, either using random values within a range defined by the practitioner or values fixed by he(she). Each generation creates a new population from the current population members $\{\boldsymbol{y}_{j,g}, j = 1, \ldots, N, g = 1, 2, \ldots\}$, where j indexes the tuning parameter vector that make up the population and g indexes the generation. The new generation is obtained using differential mutation of the population members. In our research, $N = 10000$ denotes the number of simulated scenarios and $\boldsymbol{y} = (Q, k)^\top \in \mathbb{R}^2$ the tuning parameters for the lot size and the number of demand SDs. An initial mutant parameter vector $\{\boldsymbol{v}_{j,g}, j = 1, \ldots, N, g = 1, 2 \ldots\}$ is created by choosing three members of the population $y_{j_1,g}$, $y_{j_2,g}$ and $y_{j_3,g}$, at random. Then, the elements of the initial mutant parameter vector $\boldsymbol{v}_{j,g} = (v_{j_i,g})$ are generated by

$$v_{j_i,g} = y_{j_1,g} + \zeta(y_{j_2,g} - y_{j_3,g}), \quad i = 1, \ldots, k = 2,$$

where ζ is a positive scale factor whose effective value is usually less than one (usual default: $\zeta = 0.8$). After the first mutation operation, mutation continues until k mutations have been made or until certain cross-over probability (CP) in [0, 1] is less than u, where u is a random number from the uniform distribution in [0, 1]. CR controls the fraction of the tuning parameter values that are copied from the mutant and approximates the probability that a parameter value is inherited from the mutant, since at least one mutation always occurs. Mutation is applied in this way to each member of the population.

Calculations in our simulation study were performed with the aid of the `DEoptim` package, which consists of the core function `DEoptim()` whose arguments are:

- `fn`: the function to be minimized, which must have as its first argument the vector of real-valued parameters to optimize and return a scalar real result.
- `lower`, `upper`: correspond to two vectors establishing scalar real lower and upper bounds on each tuning parameter to be optimized; the ith element of the `lower` and `upper` vectors corresponds to the ith parameter; the implementation searches the global optimum of `fn` between `lower` and `upper`.
- `...`: allows the user to pass additional arguments to the function `fn`.
- `control`: a list whose default value is the return value of `DEoptim.control()`, but whose main elements are interpreted as:
  - `VTR`: specifies the global minimum of `fn` if known, or if you wish to cease optimization after having reached a certain value (default = `-Inf`);
  - `strategy`: defines the differential evolution strategy used in the optimization procedure, described in detail by Mullen et al. (2011);

– NP: the number of population members (default $= 10 \times k$ or 50);

– bs: if bs is FALSE, then every mutant is tested against a member in the previous generation, and the best value survives into the next generation; if bs is TRUE, then the old generation and NP mutants are sorted by their associated objective function values, and the best NP vector proceeds into the next generation (default = FALSE);

– itermax: the maximum number of iterations (that is, population generations) to be allowed (default = 200);

– CR: cross-over probability from the interval [0, 1] (default = 0.9);

– F: stepsize from interval [0, 2], in our case denoted by ζ (default = 0.8);

– trace, initialpop, storepopfrom, storepopfreq, checkWinner, avWinner: are other elements of the list control, which are described in detail by Mullen et al. (2011); see also Price et al. (2006).

The return value of the function DEoptim() is a member of the S3 class DEoptim. Members of this class have a plot and a summary that allow us to analyze the optimizer's output. In our application of the DEoptim package, decision variables Q and k were lower-bounded at zero. As regards the list of tuning parameters used in our research, the default values for the DEoptim package were considered.

2.3.4 Numerical results and discussion

The underlying idea of performing a sensitivity analysis on the testing variables related to product, demand and operational characteristics is to discriminate between groups where the three distributional assumptions led to minimal TCs. Table 2.2 summarizes the numerical experiments conducted with the R software for 10000 simulated scenarios using the DE algorithm. Note that the BS distribution yields, on average, smaller TCs, EOQs, SSs, and therefore, smaller inventory levels, in comparison to the normal and gamma assumptions.

Table 2.2: summary of the simulations for the mentioned distribution and management indicator.

	Distribution		
Indicator	Gamma	Normal	BS
Sum of TCs (in \$)	205097.63	209175.49	202134.31
Sum of EOQs	767449.00	764816.00	762843.00
Sum of SSs	41166.66	53425.51	38103.58
Sum of average inventory level	424891.16	435833.51	419525.08

Table 2.3 depicts the adequacy of the distributional assumption built upon the empirical evidence presented in Silver et al. (1998). We conclude that the BS assumption for the LTD was more adequate in 81.60% of cases in terms of TCs and inventory levels; moreover, it prevails in circumstances of a higher LTD CV in comparison to gamma and normal assumptions, that is, a CV of 0.56 for the BS distribution against CVs of 0.44 and 0.34 for the gamma and normal distributions,

respectively. It is worth mentioning that, due to the numeric integral for computing the BS CDF, which diverged in less than 1% of cases, the gamma distribution was a preferable assumption in terms of TCs and inventory levels in such cases.

Table 2.3: number and percentage of times that the indicated distribution yielded minimal TC for the simulations.

Distribution	Number	%	Average CV of the LTD
BS	8160	81.60	0.56
Normal	1780	17.80	0.34
Gamma*	60	0.60	0.44

* Number of cases where numeric integral for computing the BS CDF diverged.

This chapter it is perceived that conforming to a uniform distribution suggests a predisposition to better optimize symmetric distributions and not so much positive skewed distributions. In the previous comment, we can say that when using the equation (3.15) function (function of scarcity) it is perceived that distributions such as Gaussian and BS, haveless a function of scarcity than the gamma distribution when entered in the ”Optim” function, being for the BS near zero.

Chapter 3

Inventory management for new products with triangularly distributed demand and lead-time

3.1 Introduction

Studying uncertainty of demand during lead-time or lead-time demand (LTD) is a key aspect not only for retailing and manufacturing, but also for supply chain planning (Gjerdrum et al., 2005). This uncertainty is present because demand per unit time (DPUT) and lead-time (LT) usually occur in a stochastic fashion. Therefore, DPUT, LT and LTD are random variables (RVs) following statistical distributions, which can be characterized by their corresponding probability density functions (PDFs).

We assume the LTD distribution corresponds to a sum of independent RVs, that is, an uncorrelated demand time series. The PDF of the LTD distribution is useful to determine the components of probabilistic inventory models. A model that is often used for inventory supply planning is the $(Q,\, r)$ model, which is based on the order quantity or lot size (Q) and reorder point (r). Note that Q corresponds to the quantity to be ordered when the stock achieves a certain amount of products r. The reorder point often includes a safety stock (SS) corresponding to a buffer stock used to mitigate the risk of a stock-out. The model components Q and r must be determined to minimize the total cost of the inventory management. Such a cost is function of the holding, ordering and shortage costs. When calculating the reorder point for a fixed service level, the LTD PDF is used. When the LTD distribution is unknown, this PDF can be approximated by any suitable approach. We employ a simultaneous approach to optimize Q and r; see also Silver et al. (2002).

The Gaussian (or normal) distribution is often employed to describe the RVs DPUT, LT and LTD due to its attractive properties. However, assuming normality is not always suitable to model these RVs (Lau and Lau, 2003; Rojas et al., 2015; Wanke, 2008c). When historical DPUT data for a single-product are available, a pool of non-normal distributions can be considered as candidates for modeling these data. To obtain the inventory management model, the suitable DPUT distribution must be selected by standard goodness-of-fit (GOF) methods (Barros et al., 2014). Nevertheless, there are cases where the associated LTD is difficult to obtain. Under such circumstances, empirical distributions generated from raw data may be helpful for decision making (Tersine, 1994). In the case of new products, modeling DPUT, LT and LTD is difficult because historical data are unavailable, but business decisions must be made prior to the availability of these data (Huang et al., 2015; Lariviere and Porteus, 1999). Cobb et al. (2013) studied the inventory models with a lognormal DPUT distribution and indicated the LTD distribution under different DPUT and LT distributions.

Demand uncertainty for new products has been handled by learning-based and non-learning-based approaches (Fisher et al., 2001). Under learning-based approaches, multiple production or purchasing commitments are decided first in such a way that sales data should be further obtained to update the demand forecasts and, then, to review these production/purchasing commitments (Choi et al., 2004; de Alba and Mendoza, 2001; Moe and Fader, 2002). Under non-learning-based approaches, Gaur et al. (2007) used judgmental forecasts to establish demand uncertainty, whereas Wanke (2008c) employed a uniform (UNI) distribution in that same case. Demand uncertainty of new products could also benefit from other non-learning-based approaches, such as approximations of untractable LTD distributions by considering tractable DPUT and LT distributions.

The triangular (TRI) distribution is tractable and known to be useful when data are unavailable, difficult to obtain or expensive to collect (Glickman and Xu, 2008). This distribution can be used for involving managers in the analytical process by considering their subjective estimates of the minimum, most likely (mode) and maximum values. According to Johnson (2002), the TRI distribution has the advantage of being intuitively plausible to practitioners. However, despite its long history dating back to Schmidt (1934), its recognition as a user-friendly tool is more recent (Kotz and van Dorp, 2004). Assuming triangularity may help managers in dealing with new products, which have no historical data, and therefore, offer no possibility of establishing analogies with similar products. Based on this assumption, managers may decide the first lot size to be ordered and the reorder point.

The objective of the present thesis is to propose a novel computational methodology for inventory management of new products. Specifically, we consider TRI distributions for modeling both DPUT and LT. In this case, the LTD distribution is unknown. Based on the (Q, r) inventory model, we need the LTD PDF to determine the components Q and r that minimize the expected inventory total cost. We provide an approach to estimate the actual PDF of the unknown LTD distribution obtained from triangularly distributed DPUT and LT by using polynomials and a mixture of truncated exponentials (MTEs). We evaluate the quality of the proposed approach with the Kullback-Leibler (KL) divergence (Kullback and Leibler, 1951) using the kernel non-parametric method to estimate the unknown LTD PDF such as Langseth et al. (2014). Then, we employ the approach to the unknown PDF for establishing a computational solution to the (Q, r) inventory model for new products optimizing the associated costs. Components Q and r are found by using the bisection method on the partial derivatives of the total cost function with expected shortages per cycle, which are studied under different scenarios (Heuts et al., 1986). Managerial implications for inventory decision-making are also addressed.

This chapter is organized as follows. In Section 3.2, we propose the novel methodology. In Section 3.3, we discuss a computational framework for this methodology and conducts simulations to evaluate its performance. In Section 3.4, we illustrate its potential with real-world data.

3.2 Methodology

In this section, we propose a methodology for inventory management of new products. We present a background on the TRI distribution, which is helpful to model both DPUT and LT, when their distributions are unknown due to data unavailability or difficulties to collect them. Then, we provide some details on LTD distributions obtained from the sum of independent RVs, which are useful for determining the LTD PDF. In addition, we approximate the unknown LTD PDF resulting from triangularly distributed DPUT and LT by using polynomials and MTEs. The LTD PDF is needed to determine Q and r when minimizing the inventory cost. Furthermore, we define the KL divergence to evaluate the quality of the proposed approximations in relation to an actual PDF, which is obtained with the kernel method described above. At last, we compute an analytical solution of the $(Q,\, r)$ model considering the polynomial approximation for the LTD PDF.

3.2.1 Triangular distribution

Let T be a continuous RV following a TRI distribution with parameters $a, b, c \in \mathbb{R}$, where a and b are the minimum and maximum values of T, respectively, and c is the mode of the distribution. This is denoted by $T \sim \text{TRI}(a, b, c)$. Then, the PDF, cumulative distribution function (CDF) and quantile function (QF) of T are, respectively, given by

$$f_T(t) = \frac{\mathrm{d}F_T(t)}{\mathrm{d}t} = \begin{cases} \frac{2(t-a)}{(b-a)(c-a)}, & \text{if } a \leq t \leq c; \\ \frac{2(b-t)}{(b-a)(b-c)}, & \text{if } c \leq t \leq b; \\ 0, & \text{otherwise;} \end{cases}$$

$$F_T(t) = \mathrm{P}(T \leq t) = \int_{-\infty}^{t} f_T(v)\mathrm{d}v = \begin{cases} 0, & \text{if } t < a; \\ \frac{(c-a)}{(c-b)}\frac{(t-a)^2}{(c-a)^2}, & \text{if } a \leq t \leq c; \\ 1 - \frac{(b-c)}{(b-a)}\frac{(b-t)^2}{(b-c)^2}, & \text{if } c \leq t \leq b; \\ 1, & \text{if } t > b; \end{cases}$$

$$F_T^{-1}(q) = \begin{cases} a + \sqrt{q(c-a)(b-a)}, & \text{if } 0 \leq q \leq (c-a)/(b-a); \\ b - \sqrt{(1-q)(b-c)(b-a)}, & \text{if } (c-a)/(b-a) \leq q \leq 1. \end{cases} \tag{3.1}$$

A random number generator for $T \sim \text{TRI}(a, b, c)$ is provided in Algorithm 2 based on (3.1).

Algorithm 2 Random number generator for the TRI distribution

1: Generate a uniform value u from $U \sim \text{UNI}(0, 1)$.
2: Set values for a, b and c of $T \sim \text{TRI}(a, b, c)$;
3: Compute a random number $t = t_1$ or $t = t_2$ from $T \sim \text{TRI}(a, b, c)$ using (3.1), that is,
  3.1: If $0 \leq u \leq (c-a)/(b-a)$, then $t_1 = a + \sqrt{u(c-a)(b-a)}$;
  3.2: Else $t_2 = b - \sqrt{(1-u)(b-c)(b-a)}$;
4: Repeat Steps 1 to 3 until the required number of LTD observations has been generated.

The mean and variance of $T \sim \text{TRI}(a, b, c)$ are, respectively, given by

$$\lambda = \mathrm{E}(T) = \frac{a+b+c}{3}, \quad \sigma^2 = \mathrm{Var}(T) = \frac{(b-a)^2}{18}\left(1 - \frac{(c-a)(b-c)}{(b-a)^2}\right). \tag{3.2}$$

3.2.2 Demand distribution during lead-time

Let X be a RV corresponding to the DPUT, which has mean $\mathrm{E}(X) = \lambda_X$ and variance $\mathrm{Var}(X) = \sigma_X^2$. In addition, let the RV L be the LT between the ordering of a product and its delivery, which has mean $\mathrm{E}(L) = \lambda_L$ and variance $\mathrm{Var}(L) = \sigma_L^2$. Furthermore, L is assumed to be independent from each element of the sequence of independent identically distributed RVs $\{X_1, X_2, \ldots, X_L\}$ obtained from the RV X. Moreover, assume that orders do not cross (Hayya et al., 2008). Therefore, the LTD for a product is the random sum given by

$$Y = X_1 + X_2 + \cdots + X_L, \tag{3.3}$$

with PDF $f_Y(\cdot)$ defined on $[0, \infty)$ (non-negative support), CDF

$$F_Y(y) = \int_0^y f_Y(v)\,\mathrm{d}v, \tag{3.4}$$

and QF $y(q) = F_Y^{-1}(q)$, for $0 < q < 1$. The expectation and variance of Y are, respectively, expressed as

$$\mathrm{E}(Y) = \mathrm{E}(L)\mathrm{E}(X) = \lambda_L \lambda_X, \tag{3.5}$$

$$\mathrm{Var}(Y) = \mathrm{Var}(L)(\mathrm{E}(X))^2 + \mathrm{E}(L)\mathrm{Var}(X) = \sigma_L^2 \lambda_X^2 + \lambda_L \sigma_X^2. \tag{3.6}$$

Note that, in general, the LT and DPUT can be modeled by any discrete or continuous distribution. However, since in this work we assume that managers are planning to order the first lot size and reorder point for a new product, we consider TRI distributions for both LT and DPUT.

3.2.3 Kernel estimation

By fixing minimum, maximum and mode values for the RVs LT and DPUT with TRI distributions, using Algorithm 2 to generate LT and DPUT data, and the expression given in (3.3), we are able to generate a sequence $\{y_1, \ldots, y_n\}$ of n LTD observations (data). Then, based on this sequence, we can define a kernel estimate of the unknown PDF $f_Y(\cdot)$ by

$$\widehat{f}_Y(y) = \frac{1}{n\,h} \sum_{i=1}^{n} K\left(\frac{y_i - y}{h}\right),\ y > 0, \tag{3.7}$$

where $K(\cdot)$ is a kernel function satisfying $\int_0^\infty K(y)\mathrm{d}y = 1$, h a smoothing parameter (or bandwidth) and y the point at which the PDF is estimated. The Gaussian kernel with support in $\mathbb{R}$ is often assumed for $K(\cdot)$ given in (3.7). However, we are modeling demand data with support in $[a, b]$. Thus, instead of the Gaussian kernel, it seems more natural to estimate the unknown PDF with a TRI kernel by using

$$\widehat{f}_Y(y) = \frac{1}{n} \sum_{i=1}^{n} K_{h,y}(y_i), \tag{3.8}$$

where $K_{h,y}$ is a TRI kernel of parameters h (bandwidth) and y (point at which the PDF is estimated); see details in Marchant et al. (2013).

3.2.4 Density approximations

As mentioned, we consider TRI distributions to model both DPUT and LT. Then, the LTD distribution of the product is unknown. We approximate the LTD PDF $f_Y(\cdot)$ with some suitable functions by using the PDF estimated from the kernel method as the actual LTD PDF. Consider real-valued basis functions given by

$$g_k(y) = \sum_{i=0}^{k} \alpha_i \Psi_i(y), \tag{3.9}$$

where $\alpha_i \in \mathbb{R}$ are coefficients of the function $\Psi_i(\cdot)$, for $i = 0, 1, \ldots, k$. Particular cases of $\Psi_i(\cdot)$ defined in (3.9) correspond to polynomial and exponential functions (Langseth et al., 2014). In order to approximate the PDF $f_Y(\cdot)$ from (3.9), the polynomial function of order k given by

$$\widetilde{f}_Y(y) = \sum_{i=0}^{k} \alpha_i y^i \tag{3.10}$$

may be considered. Thus, the LTD CDF given in (3.4) is approximated from (3.10) by

$$\widetilde{F}_Y(y) = \int_0^y \sum_{i=0}^{k} \alpha_i v^i \, \mathrm{d}v.$$

In addition, also from (3.9), a second approximation for the PDF $f_Y(\cdot)$ can be established by a function of MTEs with k terms as

$$\overline{f}_{Y,j}(y) = \alpha_{1h} + \sum_{i=1}^{k} \alpha_{2i,j} \exp(\alpha_{2i+1,j}\, y), \quad j = 1, \ldots, m, \tag{3.11}$$

where j corresponds to each m-piece, k-term interval; readers are referred to Rumí et al. (2006) for more details about approximation (3.11). Thus, the LTD CDF is obtained from (3.11) by

$$\overline{F}_{Y,j}(y) = \int_0^y \Big(\alpha_{1k} + \sum_{i=1}^{k} \alpha_{2i,j} \exp(\alpha_{2i+1,j}\, v)\Big) \, \mathrm{d}v, \quad j = 1, \ldots, m.$$

3.2.5 Evaluation of the approximation

To evaluate the quality of the approximation provided in (3.10), we use the KL divergence (Cobb, 2004) given in general by

$$\text{KL} = \int_{-\infty}^{\infty} \log(f(y)/\widetilde{f}(y)) f(y) \mathrm{d}y, \tag{3.12}$$

where $f(\cdot)$ is an actual PDF and $\widetilde{f}(\cdot)$ its approximation.

Such as in Langseth et al. (2014), to compute the KL divergence given in (3.12) in practice, we consider a kernel estimate as actual PDF obtaining

$$\mathrm{KL} = \int_a^b \log(\widehat{f}_Y(y)/\widetilde{f}_Y(y))\widehat{f}_Y(y)\mathrm{d}y, \tag{3.13}$$

where $\widehat{f}_Y(\cdot)$ is the TRI kernel estimate given in (3.8) and $\widetilde{f}_Y(\cdot)$ the approximation provided in (3.10) (or equivalently in (3.11)). We select the approximation whose KL value given (3.13) is the smallest one. We have empirically detected that the computational burden to calculate (3.13) is negligible, which is less than one second.

3.2.6 Inventory management models

The expected annual total cost of inventory assuming shortage is expressed as a sum of (i) the holding cost per product unit per year, denoted by $\mathrm{C_h}$, multiplied by the expected quantity in stock of product units; (ii) the ordering cost, denoted by $\mathrm{C_o}$, multiplied by the number of orders per year, and (iii) the penalty cost whenever there are stock-outs, denoted by $\mathrm{C_p}$, that is, the penalty cost per shortage product unit per year multiplied by the number of orders per year and by the expected quantity of shortage product units per year. We are assuming a business has demand 365 days a year. Thus, for the $(Q,\, r)$ model, the expected total cost per year is

$$\mathrm{C_T} = G(Q,\, r) = \left(\frac{Q}{2} + r - \mathrm{E}(Y)\right)\mathrm{C_h} + \frac{365\,\lambda_X}{Q}\mathrm{C_o} + S(r)\,\frac{365\,\lambda_X}{Q}\mathrm{C_p}, \tag{3.14}$$

where λ_X and $\mathrm{E}(Y)$ are defined in (3.2) and (3.5), respectively (Hadley and Whitin, 1963; Johnson and Montgomery, 1974; Silver et al., 2002). On the one hand, note that λ_X is multiplied by 365, because the total cost given in (3.14) is defined on an annual basis and λ_X on a daily basis. On the other hand, $\mathrm{E}(Y)$ is not altered, because its scope is verified within each safety inventory cycle. For the inventory total cost given in (3.14), $r - \mathrm{E}(Y) = \mathrm{SS} = k_q\sqrt{\mathrm{Var}(Y)}$, with the standard deviation (SD) of the LTD $\sqrt{\mathrm{Var}(Y)}$ being given from (3.6) and k_q the amount of SDs of the LTD or safety factor (SF) associated with a service level of $q \times 100\%$, for $0 < q < 1$. Note that k_q corresponds to the $q \times 100$th standardized quantile, often fixed at the 95th position for assuring a service level of 95%. In addition, in (3.14), $S(r)$ is the expected shortage per cycle given by

$$S(r) = \int_r^{y_{\max}} (y - r) f_Y(y)\, \mathrm{d}y, \tag{3.15}$$

where $y_{\max}$ is the maximum value of the LTD, r the already mentioned reorder point and, as also mentioned, $f_Y(\cdot)$ the LTD PDF. From (3.10), expression in (3.15) can be approximated by

$$\widetilde{S}(r) = \int_r^{y_{\max}} (y - r) \sum_{i=0}^{k} \alpha_i y^i \mathrm{d}y. \tag{3.16}$$

Hence, by following the well-known sum rule in integration –where summation and integral can be reversed–, (3.16) is given by

$$\widetilde{S}(r) = \sum_{i=0}^{k} \alpha_i \int_r^{y_{\max}} (y - r) y^i \mathrm{d}y = \sum_{i=0}^{k} \alpha_i \left(\frac{y_{\max}^{i+2} - r^{i+2}}{i+2} - \frac{r(y_{\max}^{i+1} - r^{i+1})}{i+1}\right). \tag{3.17}$$

Note in (3.17) that we integrate first, expressing this approximate expected shortage per cycle by a polynomial summation in r (the reorder point –still a decision variable) and $y_{\max}$ (the maximum value of the LTD). To minimize the expected total cost given in (3.14), we insert (3.17) into it, take derivatives of $G(Q,\, r)$ with respect to Q and r, equate both derivatives to zero and obtain the optimal values of Q and r of the inventory model with shortages from the solutions in Q to these two equations given by

$$Q_1 = \sqrt{\frac{2\lambda_X}{\mathrm{C_h}}\left(\mathrm{C_o} + \mathrm{C_p}\left(\sum_{i=0}^{k}\frac{\alpha_i(y_{\max}^{i+2} - r^{i+2})}{i+2} - \sum_{i=0}^{k}\frac{\alpha_i(y_{\max}^{i+1} - r^{i+1})}{i+1}\right)\right)}, \quad (3.18)$$

$$Q_2 = \frac{\lambda_X\,\mathrm{C_p}}{\mathrm{C_h}}\sum_{i=0}^{k}\frac{\alpha_i(y_{\max}^{i+1} - r^{i+1})}{i+1}. \quad (3.19)$$

Therefore, to find the optimal values of Q and r of the inventory model, we consider the equation

$$Q_1 - Q_2 = 0. \quad (3.20)$$

The equation given in (3.20) can be solved in r by applying the bisection method. Substituting r into (3.18) or (3.19), we obtain the optimal value of Q. A similar treatment may be applied to equations obtained in (3.16), (3.17), (3.18) and (3.19) when the MTE approximation given in (3.13) is used. We recall the approximation (3.10) or (3.11) can be selected from the KL value given (3.13) to be the smallest one. We use the bisection method because it is the simplest and most robust algorithm for finding the root of an one-dimensional continuous function within a closed interval. One of its properties is that it always converges. Also, it is preferable to the Newton-Raphson method when the function coefficients are unknown. In our case, it is not simple to find coefficients from equations (3.18) and (3.19). For details about the bisection method, see McNamee and Pa (2013).

3.3 Computational framework

In this section, we detail the steps of the methodology proposed in Section 3.2 with four algorithms. Then, we discuss a computational framework developed for implementing the four algorithms and study the performance of this methodology by means of a simulation study.

The sequence of algorithms below shows how inventory management of new products can be planned by companies using the methodology proposed in Section 3.2.

3.3.1 Computational implementation

R is a non-commercial and open source software for statistics and graphs, which can be obtained at no cost from `http://www.r-project.org`. The statistical software R is currently very popular in the international scientific community. For use of this software in inventory models, see Rojas et al. (2015) and Wanke and Leiva (2015). We implement the methodology introduced in this paper in the R software by using Algorithms 2-6. A computational framework for analyzing

Algorithm 3 Simulation of LTD data

1: Fix values a_2, b_2 and c_2 of the RV LT $L \sim \text{TRI}(a_2, b_2, c_2)$.
2: Generate one LT value l from $L \sim \text{TRI}(a_2, b_2, c_2)$ by using Algorithm 2.
3: Set values for the minimum (a_1), maximum (b_1) and mode (c_1) of the RV DPUT $X \sim \text{TRI}(a_1, b_1, c_1)$.
4: Simulate a number l of DPUT data $x_1, \ldots, x_l$ from $X \sim \text{TRI}(a_1, b_1, c_1)$ by using Algorithm 2.
5: Compute one LTD value y summing DPUT data $x_1, \ldots, x_l$ such as in expression (3.3).
6: Repeat Steps 1-5 to complete a number n of LTD data.

Algorithm 4 Kernel estimation of the LTD PDF

1: Generate n LTD data $y_1, \ldots, y_n$ by using Algorithm 3.
2: Estimate the LTD PDF with a TRI kernel by using expression (3.8) and the data $\boldsymbol{y} = (y_1, \ldots, y_n)^\top$ by means of the R code `density(`$\boldsymbol{y}$`, kernel = "triangular")`.

Algorithm 5 Approximation of the unknown LTD PDF

1: Generate n LTD data $y_1, \ldots, y_n$ with Algorithm 3.
2: Estimate the unknown LTD PDF with the kernel method by using Algorithm 4.
3: Approximate the unknown LTD PDF estimated with the kernel method by the polynomial function defined in (3.10) as follows:
   3.1: For $k = 2$ to $k = 20$ by 2, fit a polynomial function to the actual PDF and calculate the KL divergence value between it and the kernel estimate;
   3.2: Choose the value of k with the smallest KL divergence by using as stopping criteria a difference of less than 0.01% in the KL value of the fit between two consecutive values of k.
4: Use the polynomial of order k obtained in Step 3 as an approximation for the unknown LTD PDF.

Algorithm 6 Optimization of the total cost for the $(Q,\, r)$ model

1: Replace the polynomial coefficients of order k obtained by Algorithm 5 in derivatives (3.18) and (3.19).
2: Estimate λ_X and λ_L defined in (3.2) and (3.5) with DPUT and LT data sets, respectively, and then replace them in derivatives (3.18) and (3.19).
3: Fix holding (per unit) –C_h–, ordering –C_o– and penalty (per unit) –C_p– costs.
4: Insert the cost values fixed in Step 3 into derivatives (3.18) and (3.19).
5: Find optimal values of Q and r that minimize the total cost using (3.20), Steps 1-4 and the bisection method.

data using this approach is being developed by the authors in an R package whose "in progress" version is available upon request. Its more important functions are detailed in Table 3.1, whereas analogous functions can be considered for the MTE approximation. Some R packages related to statistical distributions that may be useful in inventory models are available at `http://CRAN.R-project.org` (Leiva et al., 2008; Stasinopoulos and Rigby, 2007).

3.3.2 Simulation results

First, we use Algorithm 3 (command: `data <- LTD(10000, tD, tLT)`) and the Monte Carlo method to simulate data in 9 different scenarios for the RVs DPUT $X \sim \text{TRI}(a_1, b_1, c_1)$ and LT $L \sim \text{TRI}(a_2, b_2, c_2)$. Each of these nine scenarios is a combination of $a_1, a_2 \in \{0.25, 0.5, 0.75\}$

Table 3.1: basic functions of an R package for the proposed methodology.

Function usage	Arguments	Description
LTD(Nsim,tD,tLT)	Nsim: number of simulated data tD: TRI(a1,b1,c1) DPUT distribution tLT: TRI(a2,b2,c2) LT distribution	It simulates LTD data from TRI distributions.
kernelTRI(d)	d: LTD data obtained from the function LTD()	It returns the kernel estimate of the PDF.
appPoly(d)	d: LTD data obtained from the function LTD()	It approximates the LTD PDF by a polynomial.
appMTE(d)	d: LTD data obtained from the function LTD()	It approximates the LTD PDF by an MTE.
KL(f,g)	f: the PDF estimated by TRI kernel g: the approximate PDF	It calculates KL divergence of g compared to f.
optimTC(kPOL,De, Ch,Co,Cp)	kPOL: vector obtained from the function appPoly() De: annual demand rate for a product Ch: holding cost Co: ordering cost Cp: penalty cost	It calculates the inventory total cost.
printReport(Report, ltd)	Report: an environment containing all the information of an evaluated scenario ltd: LTD	It prints report for a given scenario.

and $b_1, b_2 \in \{1.25, 1.5, 1.75\}$, which both are multiples of $c_1 = \lambda_X$ and $c_2 = \lambda_L$. Table 3.2 summarizes each TRI distribution parameter and the corresponding scenario position presented in Figure 3.1. In this figure, first row represents scenarios with negative skewness for DPUT distribution; second row sketches scenarios with no skewness for the DPUT distribution; whereas third row displays scenarios with positive skewness for the DPUT distribution.

Second, with the $n = 10000$ LTD data generated with Algorithm 3, we now use Algorithm 4 (command: `kernelTRI(data$LTD)`) to estimate the unknown LTD PDF with the kernel method for each scenario. Then, by using Algorithm 5, we approximate the estimated PDF with the best polynomial function (command: `appPoly(data$LTD)`) determined by the KL divergence (command: `KL(f,g)`); see Figure 3.2. From Figure 3.1, note that an excellent agreement exists between the kernel method and the approximation based on TRI distributions, but the quality of the approximation decreases as the order of the polynomial decreases, as expected. Although the polynomial adjustment for $k = 2$ does not present the smallest KL values, the real roots of the second order polynomial function are considered as proxies for the integration limits of the respective polynomial function. We also perform a robustness analysis on these results, by comparing the polynomial fit with the MTE approximation (command: `appMTE(data$LTD)`). The MTE approximation provides an average KL value of 0.0419 and a maximum KL value of 0.1400 for scenario 1; see Table 3.3. These values reflect the approximation method of Cobb et al. (2013) with four intervals; see Figure 3.4. Detailed results for the MTE approximation are omitted here due to restrictions of space, but they are available under request. Comparing the achieved KL results for both polynomial and MTE approximations, the provided approach presents better results for the polynomial approximation. Therefore, we decide to adopt it for our empirical illustration.

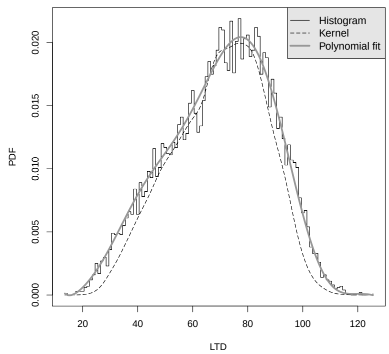
Table 3.2: summary of TRI distribution parameters for simulation scenarios.

Scenario	Row	Column	a_1	b_1	c_1	a_2	b_2	c_2
1	1	1	2.5	12.5	10	2.5	12.5	10
2	1	2	5.0	15.0	10	2.5	12.5	10
3	1	3	7.5	17.5	10	2.5	12.5	10
4	2	1	2.5	12.5	10	5.0	15.0	10
5	2	2	5.0	15.0	10	5.0	15.0	10
6	2	3	7.5	17.5	10	5.0	15.0	10
7	3	1	2.5	12.5	10	7.5	17.5	10
8	3	2	5.0	15.0	10	7.5	17.5	10
9	3	3	7.5	17.5	10	7.5	17.5	10

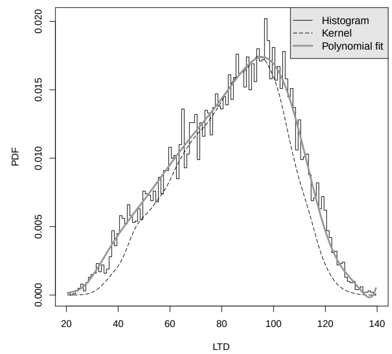
Table 3.3: summary of the best fit degree, polynomial coefficients, KL, Q and r for the indicated scenario.

Scenario	1	2	3	4	5	6	7	8	9
Degree	11	11	10	8	10	10	12	12	9
	polynomial coefficients								
α_0	0.0723	-0.4400	0.5177	0.4435	-0.7338	2.4653	37.8632	90.4228	-41.8525
α_1	-0.0194	0.0975	-0.0801	-0.0547	0.1460	-0.2226	-4.7841	-9.4490	2.8471
α_2	0.0022	-0.0093	0.0053	0.0029	-0.0106	0.0087	0.2674	0.4382	-0.0842
α_3	-0.0001	0.0005	-0.0002	-8.35E-05	0.0004	-0.0002	-0.00871	-0.0119	0.0014
α_4	6.03E-06	-1.72E-05	4.66E-06	1.47E-06	-9.51E-06	2.61E-06	0.0002	0.0002	-1.51E-05
α_5	-1.63E-07	3.94E-07	-7.17E-08	-1.60E-08	1.42E-07	-2.31E-08	-2.63E-06	-2.53E-06	1.04E-07
α_6	2.94E-09	-6.15E-09	7.32E-10	1.03E-10	-1.41E-09	1.32E-10	2.60E-08	2.11E-08	-4.73E-10
α_7	-3.55E-11	6.59E-11	-4.92E-12	-3.67E-13	9.10E-12	-4.75E-13	-1.77E-10	-1.22E-10	1.36E-12
α_8	2.81E-13	-4.75E-13	2.08E-14	5.47E-16	-3.71E-14	9.78E-16	8.04E-13	4.73E-13	-2.23E-15
α_9	-1.40E-15	2.20E-15	-5.02E-17	-	8.68E-17	-9.03E-19	-2.26E-15	-1.14E-15	1.60E-18
α_{10}	4.00E-18	-5.91E-18	5.27E-20	-	-8.85E-20	6.81E-23	3.16E-18	1.37E-18	-
α_{11}	-4.94E-21	6.99E-21	-	-	-	-	0	0	-
α_{12}	-	-	-	-	-	-	-3.94E-24	-1.28E-24	-
Q	260.4	281.1	284.8	278.2	304.6	284.8	283.4	285.0	403.6
r	94.7	108.7	112.0	105.8	121.8	112.1	110.7	112.4	170.0
	polynomial approximation								
KL	0.0232	0.0314	0.0282	0.0202	0.0237	0.0217	0.0293	0.0281	0.0366
	MTE approximation								
KL	0.1400	0.0509	0.0127	0.0375	0.0231	0.0203	0.0323	0.0475	0.0132

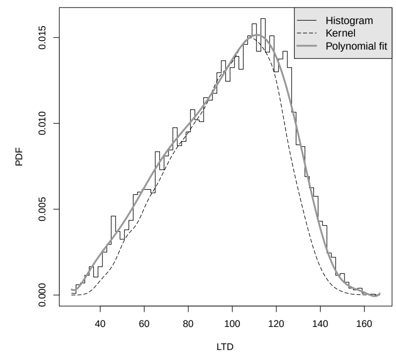
Third, we replace the polynomials of order k obtained with Algorithm 5 in the expressions of the inventory total cost derivatives with shortages given in (3.18) and (3.19). In addition, we consider (i) a capital cost of 24% per year and a cost of one product unit of $\$47$; (ii) an opportunity (holding) cost per product unit per year of $C_h = 0.24 \times \$47 = \$11.28$; (iii) an ordering cost (for placing each order) of $C_o = \$25$; and (iv) a penalty cost per shortage product unit per year of $C_p = \$10$. With this, we have the total cost function to be optimized. Now, we use Algorithm 6 (command: `optimal <- optimTC(kPOL,De,Ch,Co,Cp)`) to obtain optimal values of the (Q, r) model by applying the bisection method for each of the nine scenarios, which allows us to find the roots of the equation given in (3.20) after inserting the respective polynomial coefficients.



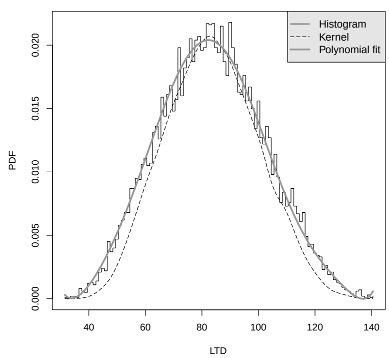
(a) Scenario 1



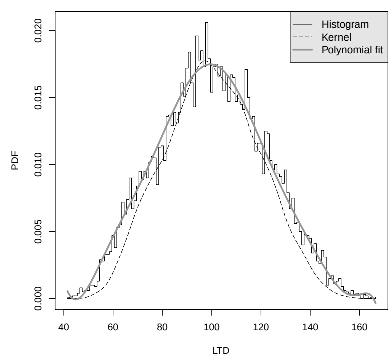
(b) Scenario 2



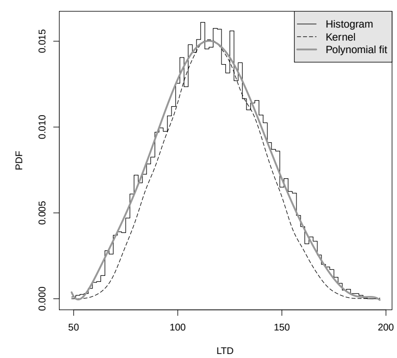
(c) Scenario 3



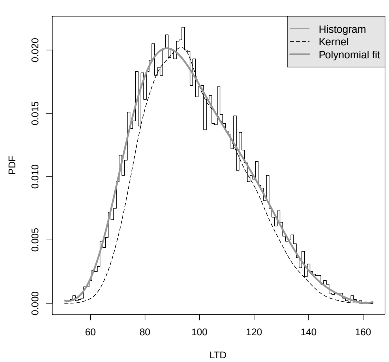
(d) Scenario 4



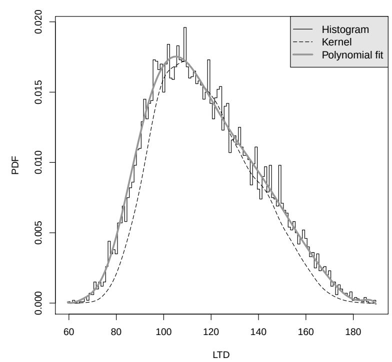
(e) Scenario 5



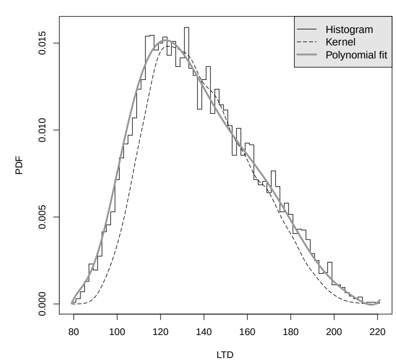
(f) Scenario 6



(g) Scenario 7



(h) Scenario 8



(i) Scenario 9

Figure 3.1: histogram, kernel estimate and polynomial fit for the LTD PDF in the indicated scenario.

The optimal r values are depicted in Figure 3.9 (command: `printReport(optimal,ltd)`). The final results for each of the nine scenarios considered are presented in Table 3.3. As expected, the skewness of the resulting LTD distribution significantly impacts the optimal Q and r values, although none of the polynomial fits provides KL values above 0.04. The tested scenarios show that this computational solution is robust enough to handle with strong asymmetries in both DPUT

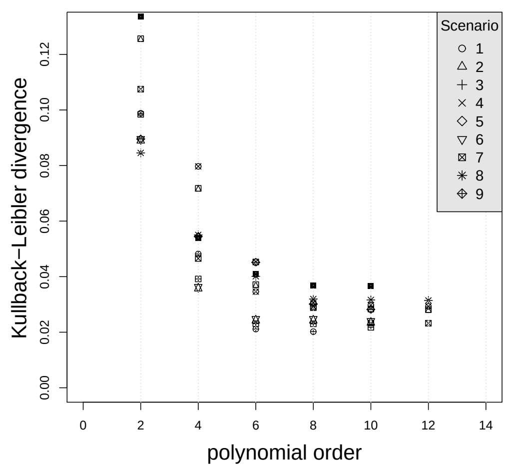


Figure 3.2: KL analysis for the polynomial fit in the indicated scenario.

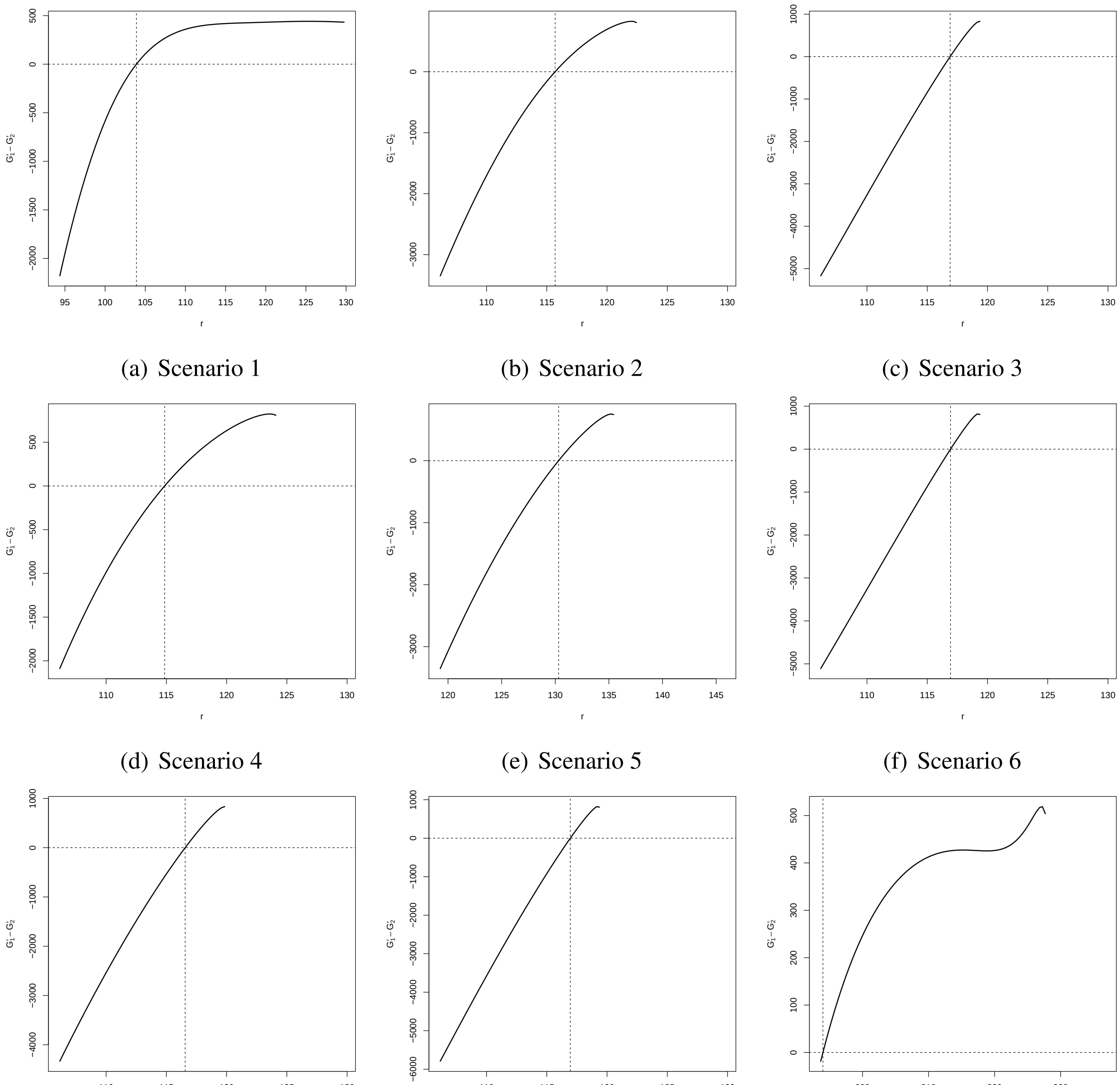
and LT distributions.

3.4 A real-world empirical illustration

In this section, we illustrate potential applications of the approach provided in Section 3.2 by using a real-world problem and the computational framework discussed in Section 3.3. First, we deal with the inventory management of new products by employing the proposed methodology considering DPUT and LT following TRI distributions. Second, we consider the inventory management of new products with (i) standard and (ii) equivalent product methodologies, both analyzing real-world demand data. Standard methodology consists of a known demand data analysis of a new product, with data collected one year after its launching. Equivalent methodology consists of a known demand data analysis for an equivalent product, with data collected one year before its launching.

3.4.1 Description of the problem

The drug supply in pharmacy units of Chilean primary health centers is channeled through their central warehouse, which acts as an intermediary between suppliers and output units (OU). The OUs receive the demand for drugs, including its own pharmacy, which performs dispensing of prescriptions to patients. This warehouse needs the storage, conservation and distribution of such drugs. Supply of warehouse is carried out by different suppliers, each of them with different delivery periods. The suppliers are selected according to technical criteria or direct negotiation, based on the needs for that time (MINSAL, 2014), which produces LT uncertainty. The warehouse delivers products on a weekly basis to all OUs by using aggregated demand requirements for each of them in the same period. When introducing a new product to the therapeutic arsenal, a problem is generated because the behavior of demand in OUs is unknown, making the determination of the



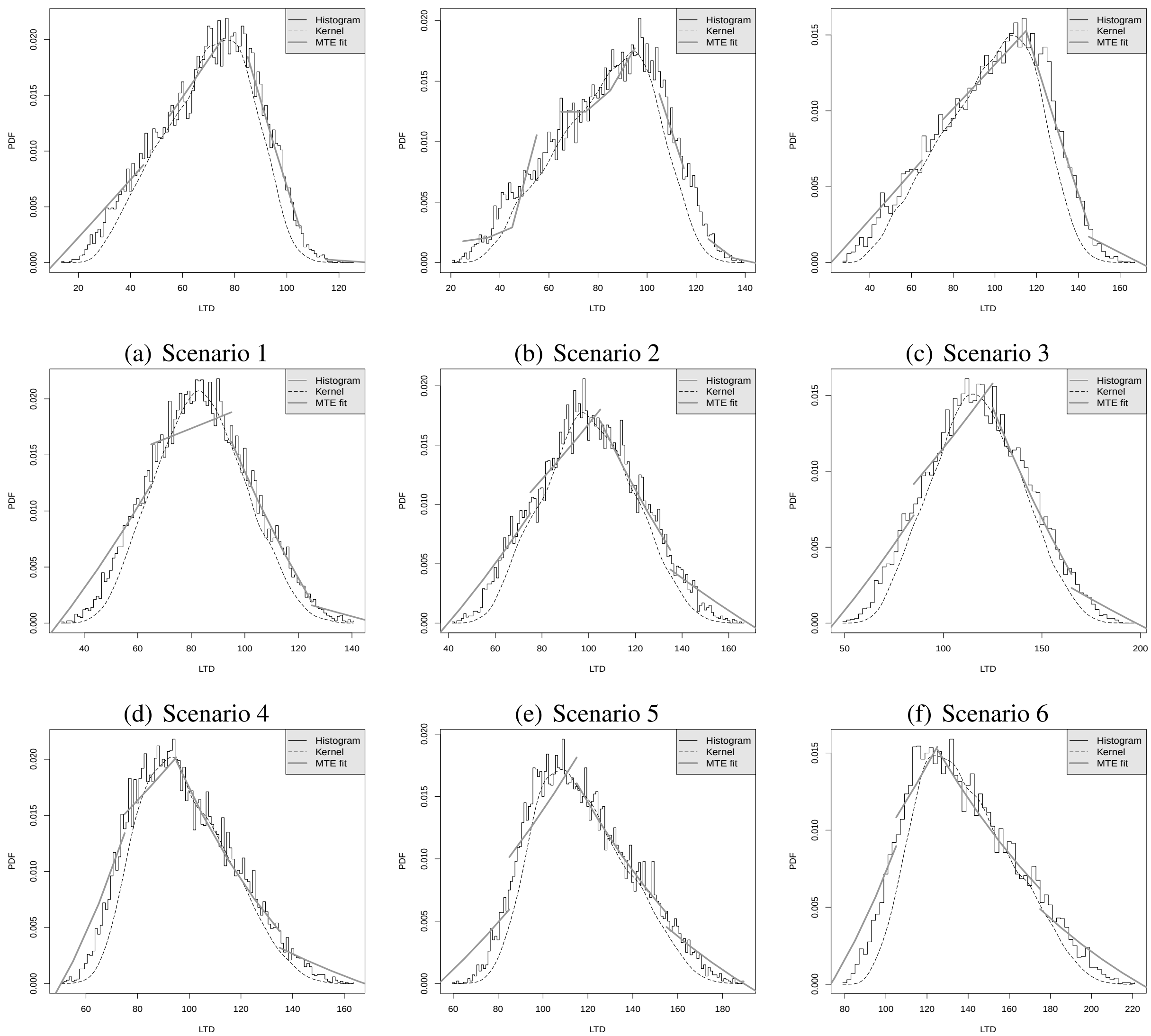
(a) Scenario 1 (b) Scenario 2 (c) Scenario 3

(d) Scenario 4 (e) Scenario 5 (f) Scenario 6

(g) Scenario 7 (h) Scenario 8 (i) Scenario 9

Figure 3.3: optimal value of r for the $(Q,\, r)$ inventory model in the indicated scenario.

lot sizes to be ordered difficult for an adequate supply. Empirically, comparisons of the behavior of aggregated demand of OUs are done based on products of similar therapeutic characteristics (equivalent) in previous periods, in order to determine the rate of demand and forecast lot sizes that satisfy the requirements. For details illustrating the supply system, see Figure 3.5. We consider the weekly actual demand of two pharmaceutical products, whose unit of measurement per coated tablet is 50 mg. One of them is an innovative product named Losartan Potassium, whose unit of



(a) Scenario 1 (b) Scenario 2 (c) Scenario 3

(d) Scenario 4 (e) Scenario 5 (f) Scenario 6

(g) Scenario 7 (h) Scenario 8 (i) Scenario 9

Figure 3.4: histograms and MTE fit for the PDF in the indicated scenario.

measurement per coated tablet is 50 mg. This pharmaceutical product replaces a similar therapeutic use product named Acetyl Salicylic Acid, whose unit of measurement per tablet is 100 mg. We consider a family health center located at the city of Concon, Chile, whose data were collected for a study of supply policy conducted by Fernando Rojas in the University of Valparaíso, Chile, during 52 weeks of the year 2012 (from January 1 to December 31). The products are shipped from the warehouse to this family health center.

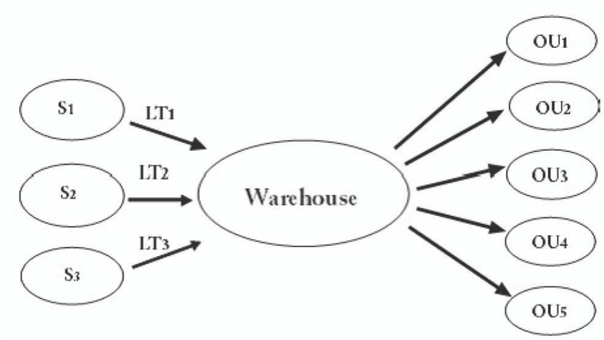


Figure 3.5: supply system in pharmacy units of primary health care centers in Chile, where S_i = supplier i, LT_i = LT of S_i for $i = 1, 2, 3$, and OU_j = output unit j, for $j = 1, \ldots, 5$.

3.4.2 The proposed methodology

By using an expert judgment for the new product Losartan Potassium, we consider the TRI DPUT parameters to be $a_1 = 0$, $b_1 = 45000$ and $c_1 = 10000$, and the TRI LT parameters to be $a_2 = 1$, $b_2 = 4$ and $c_2 = 2$. Therefore, the mean DPUT of the new product is $\lambda_X = 18333.33$ units and the mean LT of the new product is $\lambda_L = 2.33$ weeks. In addition, we consider $C_h = \$0.22$ per product unit per year, $C_o = \$138603.84$ per order and $C_p = \$9.44$ per shortage product unit per year. All of these values were provided by the manager of pharmacy unit of the primary health care center. Thus, based on the proposed methodology, the optimal values for the (Q, r) model of the new product obtained from the expected total cost per year given in (3.14) are $Q^\star = 885187$ units and $r^\star = 66917$ units, whereas the optimum total cost is $\$189343.90$.

3.4.3 The standard methodology

Now, assume a data set for the weekly actual demand of the pharmaceutical new product (Losartan Potassium in units of 50 mg), which were collected one year after its launching and are presented in Table 3.4. Table 3.5 provides some descriptive statistics of the DPUT data for the new product, such as the sample size (n), minimum and maximum values, median, mean ($\bar{x}$), SD and the coefficients of variation (CV), skewness or asymmetry (CS) and kurtosis (CK). Note that the DPUT distribution of the new product, with data collected one year before its launching, has positive skewness and moderate kurtosis. Figures 3.6(a) shows the histogram of these data with estimated gamma PDF. From this figure, note that the gamma distribution reproduces the shape of the empirical distribution of the DPUT very well and it is consistent with results presented in Table 3.6. Thus, for the standard methodology, we use a gamma DPUT distribution and a TRI LT distribution, and then we approximate the LTD distribution according to the procedure proposed by Silver et al. (2002). In addition, by considering the costs C_h, C_o and C_p, the optimal values for the (Q, r) model of the product, obtained from the expected total cost per year given in (3.14), are $Q^\star = 983435$ units and $r^\star = 138608$ units, whereas the optimum total cost is $\$247091.80$.

3.4.4 The equivalent product methodology

As usual in practice when data for new products are unavailable, data from an equivalent product can be assumed. We use data of weekly actual demand of Acetyl Salicylic Acid (in units of 100 mg), which were collected one year before the launching of the new product and are presented in Table 3.4. Table 3.5 provides some descriptive statistics of the DPUT data for the equivalent product. Note that the DPUT distribution of the equivalent product is also positively skewed and has a high kurtosis. Figure 3.6(b) shows the histogram of the equivalent product DPUT data with estimated gamma PDF. From this figure, note that the gamma distribution reproduces the shape of the empirical distribution of the DPUT fairly and it has the best fit to DPUT data. For the equivalent product methodology, we proceed analogously as in the standard methodology, obtaining the optimal values for the (Q, r) model to be $Q^\star = 1064812$ units, $r^\star = 205733$ units and an optimum total cost of $\$279537.70$.

Table 3.4: DPUT data from new (used with the standard methodology) and equivalent products.

	DPUT			DPUT			DPUT			DPUT	
Week	Standard	Equiv	Week	Standard	Equiv	Week	Standard	Equiv	Week	Standard	Equiv
1	10000	10000	14	20000	14000	27	0	14000	40	15300	10000
2	20000	45000	15	15000	21000	28	3000	7000	41	39000	20000
3	0	0	16	0	10000	29	10000	7000	42	7000	0
4	10000	0	17	15000	10000	30	5000	1200	43	30000	35000
5	0	0	18	0	11000	31	0	15000	44	10000	0
6	8000	0	19	46000	14000	32	24000	14000	45	24000	0
7	15000	15000	20	0	14000	33	12000	10000	46	25000	20000
8	0	0	21	0	14000	34	19000	14000	47	10000	10000
9	30000	9000	22	20000	11000	35	6000	7000	48	20000	10000
10	20000	20000	23	25000	21000	36	0	24000	49	10000	10000
11	12000	12000	24	9000	14000	37	0	10000	50	20000	10000
12	15000	14000	25	23000	14000	38	0	6000	51	15000	10000
13	14000	11000	26	0	10000	39	24000	16000	52	10000	19000

Table 3.5: descriptive statistics for the indicated DPUT data set.

Data set	Zeros	n	Minimum	Median	$\bar{x}$	SD	CV	CS	CK	Maximum
Standard	Yes	52	0	11000	12794.23	10846.60	0.85	0.70	3.26	46000
Equivalent	Yes	52	0	10500	11792.31	8490.36	0.72	1.25	6.40	45000
Standard	No	39	3000	15000	17058.97	9123.07	0.53	1.03	4.15	46000
Equivalent	No	44	1200	13000	13936.36	7410.963	0.53	2.09	8.96	45000

3.4.5 Fitting DPUT distributions and summary of results

For the standard and equivalent methodologies, Birnbaum-Saunders (Leiva et al., 2014b), exponential, gamma, Gaussian, inverse Gaussian, lognormal and Weibull distributions (Rojas et al., 2015) are fitted to DPUT data; see Johnson et al. (1994) and Kotz et al. (2010) for more details about these distributions. Based on the GOF Kolmogorov-Smirnov (KS) test, results presented in Table 3.6 suggest that the gamma distribution fits better the data in both standard and equivalent methodologies. We ratify visually this good fit of the gamma distribution to DPUT data from the histograms with estimated gamma PDF in Figure 3.6 and probability-probability (PP) plots with 95% acceptance bands in Figure 3.7; for more details about these graphical plots, see Castro-Kuriss et al. (2014).

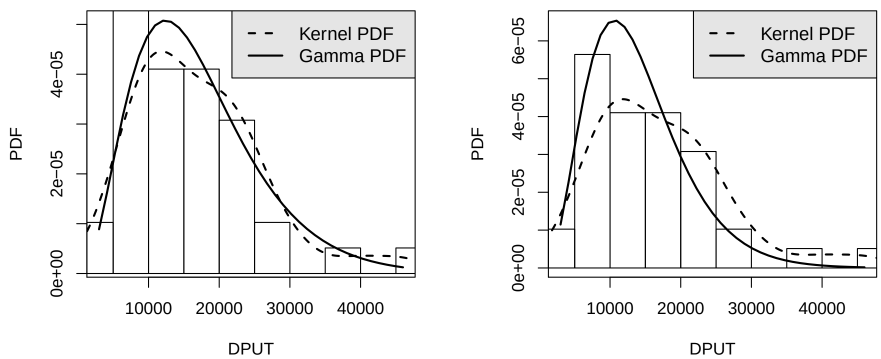


Figure 3.6: histograms with estimated PDFs from the gamma distribution and kernel method for standard (left) and equivalent (right) product DPUT data.

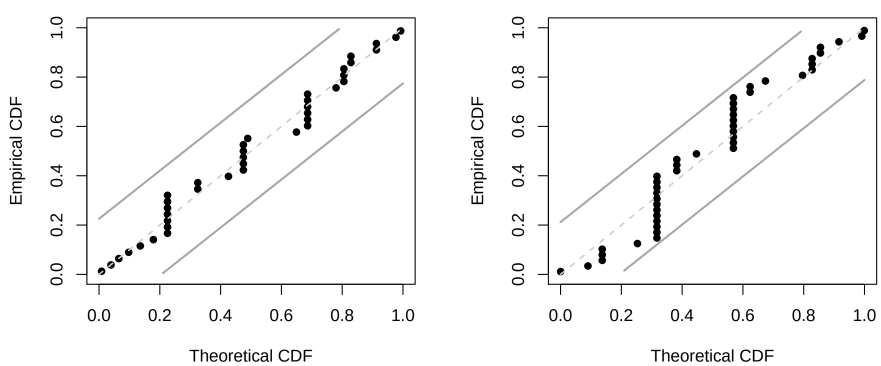


Figure 3.7: PP plots with 95% acceptance bands for standard (left) and equivalent (right) product DPUT data.

Based on each methodology, it is possible to calculate values for KL divergence, weeks on hand, turnover, lot size, expected shortage, fill rate and inventory annual total cost, as shown in Table 3.7. From this table, note that the TRI methodology produces values of the total cost, lot size and KL divergence that are closer to the standard methodology than the equivalent methodology. However,

Table 3.6: KS p-values for the indicated methodology and distribution with DPUT data.

Distribution	KS p-value (standard)	KS p-value (equivalent)
Birnbaum-Saunders	0.649	0.006
Exponential	0.003	< 0.001
Gamma	0.753	0.113
Gaussian	0.424	0.024
Inverse Gaussian	0.609	0.005
Lognormal	0.694	0.036
TRI	0.150	< 0.001
Weibull	0.823	0.086

the value of fill rate for the TRI methodology is similar to the value of the standard and equivalent methodologies.

Table 3.7: summary of results for the indicated methodology.

Methodology	DPUT distribution	KL	Weeks on hand	Turnover (per year)	Lot size	Expected shortage	Fill rate	Total cost
Proposed	TRI	0.0288	88.5	0.59	885187	23726.1	0.9732	$189343.90
Standard	Gamma	0.1225	76.9	0.68	983435	2213.2	0.9977	$247091.80
Equivalent	Gamma	0.1302	88.9	0.58	1064812	6866.5	0.9935	$279537.70

An R implementation for model inventory with random variable demand

3.4.6 Introduction

In this section, it is presented a methodology that optimizes the political system of supply and inventories through a package in R language. Initially we present the stadistic tools necessary to adjust the demand data for each distribution. Subsequently, detailed inventory models to be used for the optimization of the delivery system based on the distributions presented in Chapter 1 and 2. Then the financial indicators of our methodology is described and finally an algorithm is presented that summarizes this methodology.

Within the inventory models, there are different ways to calculate demand and LT; in this work we will see an estimate mentioned in Chapter 1 and 2 which was implemented using software R, in order to develop a methodology. Through this software, we will conduct the date and the properties of this demand, and also the estimation of the parameters of the distributions BS, gamma, IG, LN and Weibull can be carried out by the estimation of maximum likelihood estimators and goodness of fit can be accomplished through tests like Shapiro-Wilk and Kolmogorov-Smirnov (KS). Initially, the R software must be downloaded and installed (www.r-project.org) like any other software. Then Rcmdr can be used by installing the package, this package has the `Nunsummary` function that enables descriptive analysis of the data, which is used through the R editor. Finally, data analysis based on BS distribution can be performed using the packages "gbs" and "bs", for IG distribution using "ig" package and for gamma distribution, LN and Weibull through the packages "nlme", "MASS" or "gamlss". All packages must be loaded into the R software, for example, "ig" package can be installed with the command install.packages C: / Users / R / win-library / ig } having the tar.gz file copied the to directory C: / Users / R / win-library, while other packages can be downloaded directly from the menu of the form R→packages→ Install packages→Pais→ig. The databases can be entered by the `scan` function or loaded from an Excel with the function read.csv2 or from another statistics data base. The R software packages mentioned above have codes that facilitate the calculation, adjustment or parameter estimation of different distributions.

These functions enable you to use:

1. ddist () for the computation of the ML estimators and Akaike information criterion (AIC),

2. qdist () to calculate the inverse function or quantile which allows you to calculate Q from

models to perishables products and

3. rdist () for the simulation of data for a specific distribution.

An example is the case of gamma distribution range, which has the following functions; (I) dgamma, (ii) qgamma, (iii) rgamma. All these codes are integrated into the MIDA package that calculates the models mentioned in Chapter 1 and 2.

Decision variables regarding the EOQ and SS's were delineated below zero. As for the parameter tunning list used in our research, these were the defaults from the DEoptim package.

3.4.7 Simulation Study

Before applying the inventory models methodology, we should do explorative data analysis in order to define data characteristics. For this we use graphics functions, test and measurement information. To perform the exploratory analysis we use the `Nunsummary` function from the "Rcmdr" package to simulate data from a distribution.

```
set.seed (60)
```

The "set.seed" command allows us to use the same seed in future simulations, thus it can continue to have the same results, 60 is a set of random numbers which have no divergence for any of the computations of this chapter. Below is the computational code and output respectively.

```
library("MIDA")
r.bst<-rgbs(200, alpha = 11.5092, beta = 1.0275, nu = 5.1, kernel = "t")
r.bs <- rgbs(200, alpha = 1.0, beta = 1.0, kernel = "normal")
r.gamma<-rgamma(200,rate=0.5,shape=3.5)
r.norm<-rnorm(200,m=10,sd=2)
r.ig<-rinvgauss(200, 20, lambda=30)
r.ln<-rlnorm(100, 1, 0.6)
r.wei<-rweibull(200, shape=55, scale=16 )
NumSummary(muestra, statistics=c("mean", "sd", "IQR", "cv", "skewness",
"kurtosis","quantiles"))
    mean       sd      IQR       cv skewness kurtosis           0%
138.8795 487.4798 105.5403 3.510092 8.789721 95.19751 0.0003098687
       25%      50%      75%     100%   n
0.02258471 1.738911 105.5629 5823.126 200
```

The "rgbs" for example is a command generates 200 pseudorandom data similar in behavior to those obtained in the work of Rojas et al. (2015). The sample generated comes from a BS-t distribution with shape parameters at the same scale alpha = 11.5092, beta = 1.0275, nu= 5.1 degrees of freedom. From the above computational output, we can obtain the mean, standard deviation, interquartile range, coefficient of variation, skewness, kurtosis, quantile and sample size. Then we can make graphical tests for the goodness of fit tests for a BS-t distribution, using the command.

3.5 Statistical functions

A group of functions related to graphical analysis,simulation and goodness of fit for distributions is also available in this package.

```
> graficos_bst(r.bst)
> graficos_bst(r.bst)
> graficos_bs(r.bs)
> graficos_gamma(r.gamma)
> graficos_norm(r.norm)
> graficos_ig(r.ig)
> graficos_ln(r.ln)
> graficos_weibull(r.wei)
```

The adjustment date graphic `graficos_bs` is illustrated by four graphics that can be observed in Figure 3.8(a), from the histogram with the estimated FDP BS displayed in the upper left corner and a boxplot in the upper right corner, in the which can be seen positive skewness and existence atypical data, while in the lower left corner FDA (red linea) and the FDP theoretical BS (blue dots) and probability plots with envelopes or bands built by simulation processes that facilitate visual setting, with a setting of 99,954.

The data adjustment graphic `graficos_bs-t` is illustrated by four graphics that can be seen in Figure 3.8(b), from the histogram with the estimated BS-t displayed in the upper left corner FDP and boxplot in the upper right corner, in which positive skewness and the existence of atypical data can be seen, while in the lower left corner FDA (red line) and BS-t theoretical FDP (blue dots) and with envelopes probability envelope graphics or bands built by a simulation process that facilitate visual setting, with a setting of 99,323.

The data adjustment graphic `graficos_gamma` is illustrated by four graphics that can be seen in Figure 3.8(c), from the histogram with the estimated FDP range displayed in the upper left corner and a boxplot in the upper right corner, in which positive skewness and the existence of atypical data can be seen, while in the lower left corner FDA (red line) and theoretical range FDP (blue dots) and probability points with envelope graphics or bands built by a simulation process that facilitate visual setting, with a setting of 99,972.

The data adjustment graphic `graficos_ig` is illustrated by four graphics that can be seen in Figure 3.8(d), from the histogram with the estimated IG range displayed in the upper left corner and a boxplot in the upper right corner, in which positive skewness and the existence of atypical data can be seen, while in the lower left corner FDA (red lines) and the FDP theoretical IG (blue dots) and probability plots with envelopes or constructed by a process simulation that facilitates visual settings, with a setting of 99,836 bands.

The data adjustment graphic `graficos_norm` is illustrated by four graphics that can be seen in Figure 3.9(a), from the histogram with the estimated FDP range displayed in the upper left corner and a boxplot in the upper right corner, in which positive skewness and the existence of atypical data can be seen, while in the lower left corner FDA (red line) and theoretical range FDP (blue dots) and probability points with envelope graphics or bands built by a simulation process that facilitate visual setting, with a setting of 99,972.

The data adjustment graphic `graficos_ln` is illustrated by four graphics that can be seen in Figure 3.9(b), from the histogram with the FDP estimated lognormal displayed in the upper left corner and a boxplot in the upper right corner, in which positive skewness and the existence of atypical data can be seen, while in the lower left corner FDA (red lines) and the FDP theoretical lognormal (blue dots) and probability plots with envelopes or constructed by a process simulation that facilitates visual settings, with a setting of 99,964 bands.

The data adjustment graphic `graficos_weibull` is illustrated by four graphics that can be seen in Figure 3.9(c), from the histogram with the FDP estimated lognormal displayed in the upper left corner and a boxplot in the upper right corner, in which positive skewness and the existence of atypical data can be seen, while in the lower left corner FDA (red lines) and the FDP theoretical lognormal (blue dots) and probability plots with envelopes or constructed by a process simulation that facilitates visual settings, with a setting of 99,961 bands.

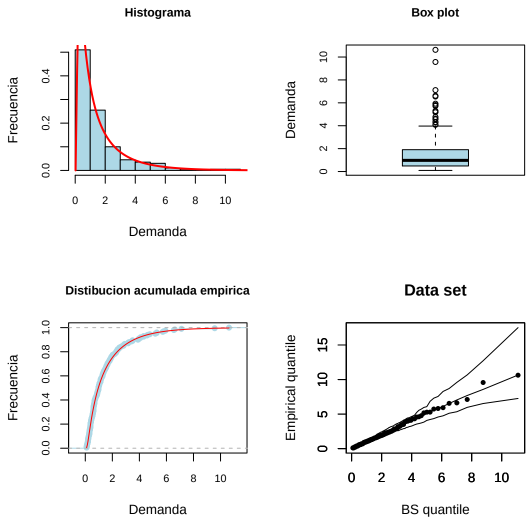
3.5.1 Goodness of fit functions

In order to determinate the KS, AIC and BIC, respectively, have been developed.

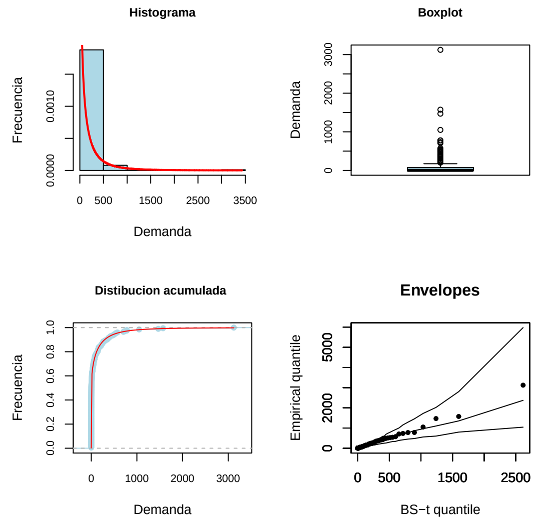
```
>ks_bst(r.bst)
0.359
>ks_bs(r.bs)
0.968
>ks_gamma(r.gamma)
0.931
>ks_ig(r.norm)
0.7323
>ks_ln(r.ig)
0.7717
>ks_wei(r.wei)
0.9696
```

In the previous computational output, we have the KS test p value is 0.36 indicating that there is not evidence to reject the hypothesis, that is, the data comes from a BS t distribution, this function is derived from the ksgbs function from the gbs package. In our KS test for BS data we have a p value of 0.97, indicating that there is not evidence to reject the hypothesis, that is, the data come from a BS distribution.

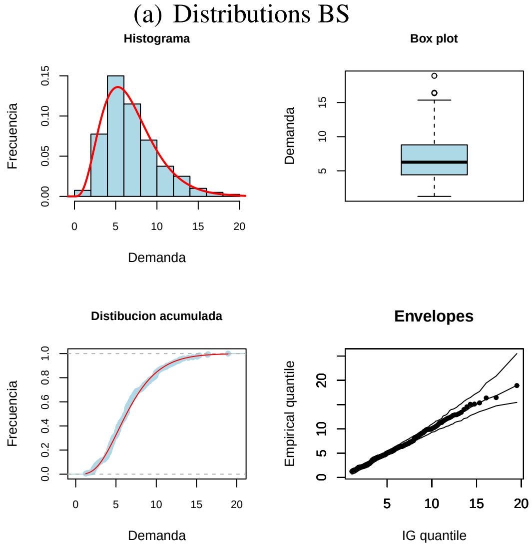
In the previous computational output the KS test output p value of 0.93 , indicating that there is not evidence to reject the hypothesis, that is, the data come from a gamma distribution, while we have the p value from the KS test as 0.73 indicating that there is not evidence to reject the hypothesis, that is, the data comes from a IG distribution, this function is derived from the “ig” package, also we have the p value from the KS test as 0.77 indicating that there is no evidence to reject the hypothesis, that is, the data fits a distribution lognormal and we have the value of p in the KS test as 0.97 indicating that there is no evidence to reject the hypothesis, ie, the data fits a Weibull distribution.



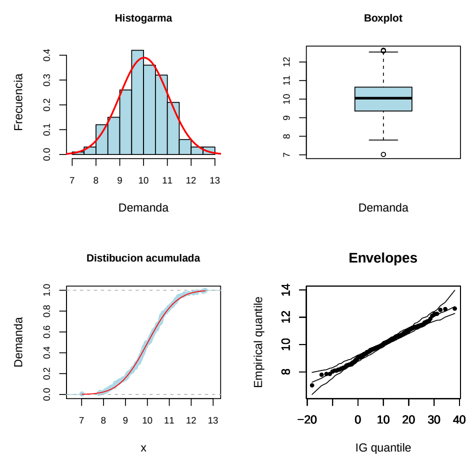
(a) Distributions BS



(b) Distributions BS-t



(c) Distributions gamma



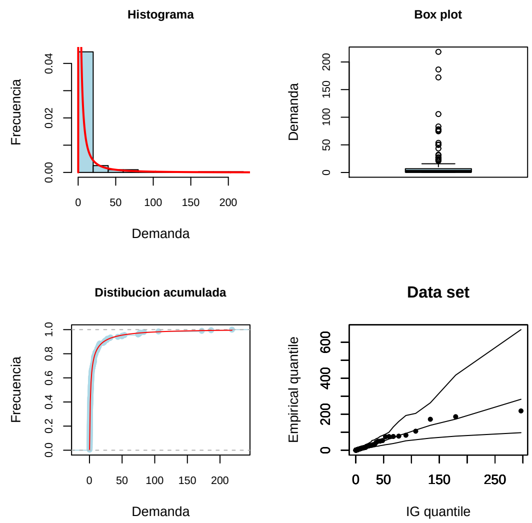
(d) Distributions inverse Gaussian

Figure 3.8: Graphics goodness of fit for distributions.

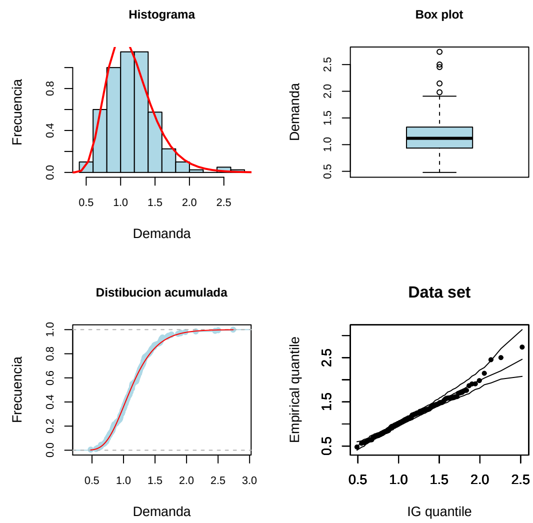
For Gaussian distributions can be generated from function `shapirop(x.norm)` with p-value as 0.59 indicating that there is no evidence to reject the hypothesis, ie, the data fits a normal distribution.

Generic function calculating Akaike information criterion, also known as AIC for one or several fitted model objects for which a log-likelihood value can be obtained, according to the formula $-2(log - likelihood) + 2npar$, where n par represents the number of parameters in the fitted model.

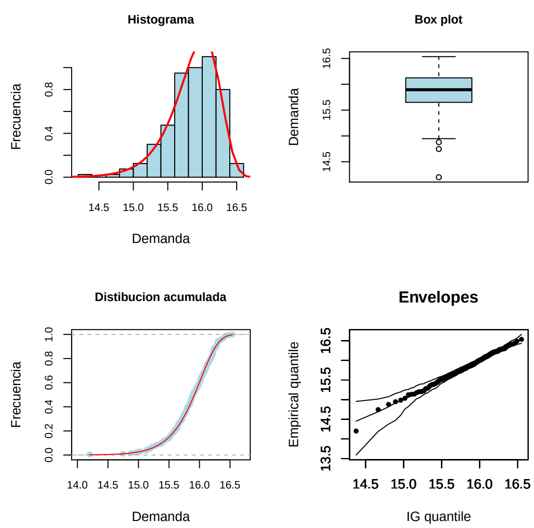
```
aic_bst(r.bst)
1223.318
```



(a) Distributions Gaussian



(b) Distributions lognormal



(c) Distributions Weibull

Figure 3.9: Graphics goodness of fit for distributions.

```
aic_bs(r.bs)
562.190
aic_gamma(r.gamma)
1078.388
aic_nor(r.bs)
865.271
aic_ig(r.ig)
970.350
aic_ln(r.ln)
```

```
163.608
aic_wei(r.wei)
140.978
```

This generic function calculates the Bayesian information criterion, also known as BIC, for one or several fitted model objects for which a log-likelihood value can be obtained, according to the formula $-2 * log - likelihood + npar * log(m)$, where n par represents the number of parameters and m the number of observations in the fitted model.

```
bic_bst(r.bst)
1229.914
bic_bs(r.bs)
568.786
bic_gamma(r.gamma)
1084.985
bic_nor(r.norm)
584.778
bic_ig(r.ig)
976.947
bic_ln(r.ln)
170.205
bic_wei(r.wei)
147.575
```

3.5.2 Model inventory functions

For model Economic Order Quantity (EOQ) without planned shortages and reorder point, we have Q is obtained.

```
EOQ_bst(r.bst,9900,2,3)
1776.928
EOQ_bs(r.bs,9900,2,3)
219.651
EOQ_gamma(r.gamma,3,0.1,3)
18.2621
EOQ_norm(r.norm,3,2,3)
9.481
EOQ_ig(r.ig,300,2,3)
84.931
EOQ_ln(r.ln,300,2,3)
9.486
EOQ_wei(u,300,1,3)
170.272
```

For the model for EOQ with demand BS-t, where OC = 9900, SC = 2 and LT = 3, indicating that the amount of articles to buy is 1777. Now consider a pseudorandom sample of simulated data from the BS distribution, we have the amount of articles to buy is 220, for EOQ with demand gamma, where OC = 3, SC = 0.1 and LT = 3, indicating that the purchase amount is 18 articles, for EOQ with demand Gaussian, where OC = 3, SC = 0.1 and LT = 3, indicating that the purchase amount is 18 articles, for EOQ with demand IG, where OC = 300, SC = 2 and LT = 3, indicating that the purchase amount is 85 products, for EOQ with demand lognormal, where OC = 300, SC = 2 and LT = 3, indicating that the amount to buy is 9 products, for EOQ with demand Weibull, where OC = 300, SC = 1 and LT = 3, indicating that the purchase amount is 170 products.

Implements the reorden point (ROP) for model EOQ, we have r is obtained:

```
ROP_gbst(r.bst,0.95,3)
531.877
ROP_bs(r.bs,0.95,3)
13.028
ROP_gamma(r.gamma,0.95,3)
34.808
ROP_nor(r.norm,0.95,3)
19.955
ROP_ig(r.ig,0.95,3)
30.819
ROP_ln(r.ln,0.95,3)
5.530
ROP_wei(r.wei,0.95,3)
48.917
```

For the model for ROP with demand BS-t, where α = 0.95 (confidence coefficient) and LT = 3, the computational output indicates that the stock level reaches 532 products are ordered, for ROP with demand BS, the computational output indicates that the stock level reaches 13 products are ordered, for ROP with demand gamma, the computational output indicates that the stock level reaches 35 products are ordered, for ROP with demand Gaussian, the computational output indicates that the stock level reaches 20 products are ordered, for ROP with demand IG, the computational output indicates that the stock level reaches 31 products are ordered, for ROP with demand lognormal, the computational output indicates that the stock level reaches 6 products are ordered, for ROP with demand Weibull, the computational output indicates that the stock level reaches 49 products are ordered.

For the model for perishables products, we have d^0 is obtained:

```
q_gbt(rbs,150,8,9)
1646.190
q_bs(r.bs,150,8,9)
23.956
q_gamma(r.gamma,40,0.1,3)
13.59036
```

```
q_norm(r.norm,90,1,4)
19.955
q_ig(r.ig,300,1,2)
109.619
q_ln(r.wei,300,1,2)
2.464
q_wei(u,300,1,2)
16.435
```

For the model for for perishables products with demand BS-t, where UC= 150, HC= 8 and PC= 9, indicating that the amount of articles to buy is 1646, for perishables products with demand BS, indicating that the purchase amount is 24 articles, for perishables products with demand gamma, where UC=40, HC=0.1 and PC=3, indicating that the purchase amount is 14 articles, for perishables products with demand Gaussian, where HC=90, HC=1 and PC=4, indicating that the purchase amount is 20 articles, for perishables products with demand IG, where UC=300, HC=1 and PC=2, indicating that the purchase amount is 110 articles,for perishables products with demand BS, where UC=80, HC=2 and PC=9, indicating that the purchase amount is 2 articles,for perishables products with demand Weibull, where UC=80, HC=2 and PC=9, indicating that the purchase amount is 24 articles.

3.5.3 Differential evolution algorithm with Optim function

The function delivers the computation time through system.time() function, which is reduced with the cmpfun function of compiler package that reduces by more than a third time taking a application time of 4970.72 sec . Then the function delivers a file called RESULTS NUMERICAL.csv which is stored in the ”My Documents” folder which contains all the parameters and costs associated to the demand models and their distributions. Then with the following code we can read the data:

```
results = read.csv2 ( "C: \\ Users \\Desktop\\ RESULTS_NUMERICAL.csv",
Header = TRUE, dec = ".", Sep = ",")
```

DEMIRAF (hol, CTR, Cip, xis, sd, lt, slt)

1. xis: Demand
2. sd: Standard deviation of demand
3. lt: lead time
4. slt: Standard deviation of de lead time
5. hol: holding costs.
6. CTR: ordering costs.
7. Cip: storage costs.

Conclusions

In the chapter 1, it show a methodology useful for food service companies that allows their contribution margins to be optimized. The methodology was based on statistical tools, inventory management models and financial indicators. Its main steps were synthesized in Algorithm 1. Because there is a need to conduct case studies focusing on their applicability in firms to reduce the gap between theory and practice, and to transfer knowledge to industry, we applied this methodology. Specifically, the case study was conducted to a Chilean food service company, which showed the importance of considering inventory models and statistical aspects for improving its supply and inventory policies, increasing its contribution margins. Inventory management models for perishable products showed to adjust adequately the demand for fruits and vegetables, which have the greatest unit contribution margins, so that such products can be considered as critical in the inventory assortment. With respect to the non-perishable products, it is noteworthy that the EOQ model fitted them considerably well. Such products can be stored indefinitely, without losses occurring through maturity, which has been used to reduce ordering costs. Products fitted by the model for perishables presented an optimized quantity similar to the demand rate, because such products have a shelf life so that they cannot be stored for a long period. A small amount of products (about 5%) used a JIT method. In summary, we validated improvements in logistics management applying an appropriate inventory model, increasing the contribution margins of a Chilean company. This result agrees with that reported by Ramanathan (2006), who linked inventory cost minimization and contribution margin maximization in products of type A of the ABC classification. These products correspond to around 20% of the total of products of the inventory assortments and are responsible for a proportion around 80% of the total contribution margin. It is noteworthy that, although we attained an improvement of 10.47% in contribution margin for the studied Chilean company, 1.8% in total variable contribution margin, 54.64% in ordering costs, and 84.05% in storing costs, using the proposed methodology, still some aspects can be improved. For example, it is possible to explore the statistical dependence among products. Seasonality and trend factors, as well as dependence in the time of the demand, can be considered in the modeling by using time series models. Statistical dependence among products can be analyzed by means of multivariate structures for the models considered in this work.

In the chapter 2, we assessed how different demand distributions during lead-time interact with relevant characteristics of the product. We considered the choice of the optimal inventory policy in terms of total costs of the inventory management. Differently from most previous studies on the topic that exclusively explored the effects of one single distributional assumption, this thesis also analyzed its adequacy in light of some inventory management key elements for decision making, such as cost, demand and lead-time. It was shown that the Birnbaum-Saunders distribution outperformed the normal and gamma assumptions with respect to demand uncertainty during the lead-time. Departing from what was found in previous studies, the obtained results provide a guidance on the selection of the most appropriate distributional assumption for the demand during the lead-time. Specifically, while the Birnbaum-Saunders distribution is more adequate to handle demand during lead-time with a high coefficients of variation, the gamma and the normal assumptions should be restricted to well-behaved patterns. This paper has also can presented a practical contribution by means of numerical analysis conducted with the aid of a computational code developed for such purposes in the R statistical software.

In the chapter 3, we proposed a methodology to deal with the inventory management of new products by using triangular distributions for both demand per unit time and lead-time. Inventory shortage and total cost expressions for the $(Q,\, r)$ model were provided and computationally implemented for triangularly distributed demand per unit time and lead-time, based on polynomial and mixture of truncated exponential approximations for the probability density function of the demand during lead-time. We evaluated our methodology under nine different scenarios in order to assess the robustness of the computational procedure. Managers, however, can easily alter the parameters of the original demand and lead-time distributions to assess any particular case. New product inventory management represents an appropriate case for applying the results derived in this paper, because demand data tend to be difficult or expensive to collect. The main advantage of assuming a triangular distribution is its ease of involving managers in the analytical process by capturing their subjective estimates in terms of the minimum, most likely and maximum values. This is particularly important when decisions regarding the first lot size and reorder point must be made. The idea is to build a consensus basis regarding minimal, most likely and maximal demand forecast for a given new product and to deploy its impacts on the inventory replenishment model. This non-learning based approach could be used as a first decision round on lot sizes and reorder points, whereas the learning process related to the distribution is still ongoing. The polynomial approximation to the triangularity assumption proved to be fairly robust, with extremely small Kullback-Leibler divergence values.

In the chapter 4, we presents a development package coded in R. This package includes four graphics that help identify the distribution of data, along with the KS and Shapiro Wilk tests of goodness of fit to verify whether the data fits a distribution proposal for the package; then, if this is set to more than one distribution, we can use various functions such as AIC and BIC in order to know what is the distribution that best fits the data; we have three inventory models with batch sizes within a variety of distribution statistics then we have the policy of using a ROP if the model requires it. One of the great strengths of the package is the autonomy of each function, where these bring incorporated the calculation of the parameters, expectation, variance and others necessary for its function in order to facilitate and accelerate their application. On the other hand, the user must be aware of the distributions, because the results are immediate and allow us to see where errors occur; a case arises as the range of the parameters of each distribution in specific, an example is the BS-t with $\nu < 4$, this renders an undefined variance, which does not allow to calculate the ROP for the model (Q, R). To develop the inventory models, an algorithm called differential evolution is incorporated which generates good results for the calculation of minimum costs and batch size, but because of the form of the E(Y) and V(Y) it could not be performed for an IG distribution, lognormal and Weibull, as their estimators possess nonlinear functions, such as the logic that impeded their adjustment.

Appendix

Algorithms in R

```
library("MASS") # fitdistr (mle)
library("bs") # BS
library("ig") # inv gauss
library("gbs") #BS
library("Rcmdr") # summary

#====== Histograma y Plots ======================
graficos_bst <- function(x, entitled = "Envelopes"){
par(mfrow=c(2,2))
minimum <- min(x) - sd(x)
maximum <- max(x) + sd(x)
axisx <- seq(minimum, maximum, by = 0.1)
mlebst(x)
alphaEst <- mlebst(x)$alpha
betaEst  <- mlebst(x)$beta
nuEst    <- mlebst(x)$nuOptimal
valuesy <-dgbs(axisx, alpha = alphaEst, beta =betaEst, nu = nuEst, kernel = "t")
hist(x, main='Histograma', col='lightblue',ylab='Frecuencia',xlab='Demanda'
, cex.main=0.9, cex.axis=0.8,prob=TRUE )
lines(axisx, valuesy, lwd = 2, col = "red")

boxplot(x, col='lightblue', main='Boxplot',ylab='Demanda', cex.main=.9, cex.axis=.8)
plot(ecdf(x),xlab='Demanda',main='Distibucion acumulada ',col='lightblue',
ylab='Frecuencia', cex.main=0.9, cex.axis=0.8,prob=TRUE)
z <- seq(min(x), max(x), (max(x)-min(x))/(length(x)-2))
lines(z,pgbs(z, alpha = alphaEst, beta = betaEst, nu = nuEst,kernel = "t"),
lty=1,col='red',)
n        <- length(x)
d2s     <- sort(x)
xq2     <- qgbs(ppoints(n), alpha = alphaEst, beta = betaEst, nu = nuEst,kernel = "t")
Xsim     <- matrix(0, 100, n)
for(i in 1:100){
u2       <- rgbs(n, alpha = alphaEst, beta = betaEst, nu = nuEst,kernel = "t")
Xsim[i,] <- u2
}
Xsim2      <- apply(Xsim, 1, sort)
```

```
d21        <- matrix(0, n, 1)
d22        <- matrix(0, n, 1)
for(i in 1:n){
d21[i]  <- quantile(Xsim2[i,], 0.025)
d22[i]  <- quantile(Xsim2[i,], 0.975)
}
d2med      <- apply(Xsim2, 1, median)
fy         <- range(d2s, d21, d22)
plot(xq2, d2s, xlab = quote("BS-t quantile"),ylab = "Empirical quantile",
pch = 20, ylim = fy)
par(new = T)
plot(xq2, d21, type = "l", ylim = fy, xlab = "", ylab = "")
par(new = T)
plot(xq2, d2med, type = "l", ylim = fy, xlab = "", ylab = "")
par(new = T)
plot(xq2, d22, type = "l", ylim = fy, xlab = "", ylab = "")
title(entitled)
Rsq <- cor(xq2, d2med)^2*100
print(paste0('Ajuste de envelopes:', Rsq))
}
ks_bst<-function(x){
g<-ksgbs(rbst, kernel = "t", graph = FALSE, xLabel = "data",yLabel = "cdf")$p.value
list(valor_p=g)
}
#=============== AIC ===================
aic_bst <- function(x){
estimates <- mlebst(x)
alpha     <- estimates$alphaEstimate
beta      <- estimates$betaEstimate
nu        <- estimates$nuOptimal
AIC    <- (-2) * log(prod(dgbs(x, alpha, beta, nu, kernel = "t"))) + 4
print(paste0(AIC))
}
aicbst(rbst)
#============= BIC ==================
bic_bst <- function(x){
estimates <- mlebst(x)# gbs
alpha     <- estimates$alphaEstimate
beta      <- estimates$betaEstimate
nu        <- estimates$nuOptimal
BIC    <- (-2) *log(prod(dgbs(x, alpha, beta, nu, kernel = "t"))) + 2 * log(length(x))
print(paste0(BIC))
}
#-----------------------------EOQ  ------------------------------#
EOQ_bst<-function(x,CL,CS,l)
{
estimates <- mlebst(x)
alpha     <- estimates$alphaEstimate
beta      <- estimates$betaEstimate
nu        <- estimates$nuOptimal
A         <- (nu/(nu-2))
lambda    <- ((l*beta)/2)*(2+ (A)*(alpha)^2)
```

```
result <- sqrt((2 *lambda * CL)/CS)
print(paste0(result))
}
#----------------------------EOQ  ------------------------------#
ROP_bst<-function(x,k,l)
{
estimates <- mlebst(x)
alpha     <- estimates$alphaEstimate
beta      <- estimates$betaEstimate
nu        <-  estimates$nuOptimal
A         <- (nu/(nu-2))
B         <- ((3*(nu^2))/((nu-4)*(nu-2)))
lambda    <- ((l*beta)/2)*(2+ (A)*(alpha)^2)
xp        <- qgbs(k, alpha, l*beta, nu, kernel="t")
sigma     <- sqrt(((((l*beta)^2)*(alpha^2))/4)*(4*(A) + (2*(B)-(A^2))*(alpha)^2))
kp    <- (xp-lambda)/sigma
result    <- (lambda) + kp*sigma
print(paste0(result))
}
#para articulos perecibles
q_bst<-function(x,CP,CS,C)
{
estimates <- mlebst(x)
alpha     <- estimates$alphaEstimate
beta      <- estimates$betaEstimate
nu        <- estimates$nuOptimal
RC<-(CP-C)/(CP+CS)
y         <- qgbs(RC, alpha, beta, nu, kernel = "t")
print(paste0(y))
}
qgbt(rbst,150,-8,9)
#------------BS------------------------------------------
graficos_bs<-function(x, entitled = "Data set")
{
par(mfrow=c(2,2))
minimum <- min(x) - sd(x)
maximum <- max(x) + sd(x)
axisx <- seq(minimum, maximum, by = 0.1)
mlebs(x)
alpha<- mlebs(x)$alphabsEstimate
beta <- mlebs(x)$betabsEstimate
valuesy <-dgbs(axisx, alpha = alpha, beta =beta, kernel = "normal")
hist(x, main='Histograma', col='lightblue',ylab='Frecuencia',xlab='Demanda',
cex.main=0.9, cex.axis=0.8,prob=TRUE )
lines(axisx, valuesy, lwd = 2, col = "red")
boxplot(x, col='lightblue', main='Box plot',ylab='Demanda', cex.main=.9, cex.axis=.8)
plot(ecdf(x),main='Distibucion acumulada empirica',col='lightblue',xlab='Demanda'
, ylab='Frecuencia',cex.main=0.9, cex.axis=0.8,prob=TRUE)
z <- seq(min(x), max(x), (max(x)-min(x))/(length(x)-2))
lines(z,pgbs(z, alpha = alpha, beta = beta, kernel = "normal"),lty=1,col='red')
xq2     <- qbs(ppoints(length(x)), alpha = alpha, beta = beta)
Xsim     <- matrix(0, 100, length(x))
```

```
for(i in 1:100){
u       <- rgbs(length(x), alpha = alpha, beta = beta)
Xsim[i,] <- u
}
Xsim2      <- apply(Xsim, 1, sort)
d21        <- matrix(0, length(x), 1)
d22        <- matrix(0, length(x), 1)
for(i in 1:length(x)){
d21[i]  <- quantile(Xsim2[i,], 0.025)
d22[i]  <- quantile(Xsim2[i,], 0.975)
}
d2med      <- apply(Xsim2, 1, median)
fy         <- range(sort(x), d21, d22)
plot(xq2, sort(x), xlab = quote("BS quantile"),
ylab = "Empirical quantile",pch = 20, ylim = fy)
par(new = T)
plot(xq2, d21, type = "l", ylim = fy, xlab = "", ylab = "")
par(new = T)
plot(xq2, d2med, type = "l", ylim = fy, xlab = "", ylab = "")
par(new = T)
plot(xq2, d22, type = "l", ylim = fy, xlab = "", ylab = "")
title(entitled)
Rsq <- cor(xq2, d2med)^2*100
print(paste0('Ajuste de envelopes:', Rsq))
}
#=========KS===========================
ks_bs<-function(x){
p<-ksbs(x, plot = T, alternative = "two.sided")$p
print(paste0(p))
}
ks_bs(rbs)
#------------AIC -------------------------
aic_bs <- function(x) {
estimates<- mlebs(x)
alpha_bs     <- estimates$alphabsEstimate
beta_bs      <- estimates$betabsEstimate
aic <- (-2)*log(prod(dbs(x,alpha_bs,beta_bs))) + 4
print(paste0(aic))
}
aic_bs(rbs)
#============ BIC ==================
bic_bs <- function(x){
estimates<- mlebs(x)
alpha_bs     <- estimates$alphabsEstimate
beta_bs      <- estimates$betabsEstimate
BIC    <- (-2)*log(prod(dbs(x,alpha_bs,beta_bs)))+ 2*log(length(x))
print(paste0(BIC))
}
bicbs(rbs)
#-----------------------------EOQ -------------------------------#
EOQ_bs<-function(x,CL,CS,l)
{
```

```
estimates <-mlebs(x)
E_BS<-estimates$betabsEstimate*(1+estimates$alphabsEstimate^2/2)
result <- sqrt((2 *l*E_BS * CL)/CS)
print(paste0(result))
}
#------------------------------ROP -------------------------------#
ROP_bs<-function(x,k, l)
{
estimates <-mlebs(x)
E_BS<-estimates$betabsEstimate*(1+estimates$alphabsEstimate^2/2)
V_BS<-estimates$betabsEstimate^2*estimates$alphabsEstimate^2*
(1+5*estimates$alphabsEstimate^2/4)
xp        <- qbs(k,alpha = estimates$alphabsEstimate,
beta =l*estimates$alphabsEstimate,log=FALSE)
kp  <- (xp-l*E_BS)/(l*sqrt(V_BS))
result<- (l*E_BS)+ kp*l*sqrt(V_BS)
print(paste0(result))
}
ROPbs(rbs,0.95,3)
#-----------------------articulos perecibles--------------------------------
q_bs<-function(x,CP,CS,C)
{
estimates <-mlebs(x)
E_BS<-estimates$betabsEstimate*(1+estimates$alphabsEstimate^2/2)
V_BS<-estimates$betabsEstimate^2*estimates$alphabsEstimate^2*
(1+5*estimates$alphabsEstimate^2/4)
RC<-(CP-C)/(CP+CS)
y <- qbs(RC,alpha = mleig(x)$muEstimate, beta =mleig(x)$lambdaEstimate,log=FALSE)
print(paste0(y))
}
#------------Gamma--------------------------------------------
graficos_gamma<-function(x, entitled = "Envelopes")
{
par(mfrow=c(2,2))
minimum <- min(x) - sd(x)
maximum <- max(x) + sd(x)
axisx <- seq(minimum, maximum, by = 0.1)
ajuste.gamma <-fitdistr(x,"gamma")
valuesy <- dgamma(axisx,shape=ajuste.gamma$estimate[1],rate=ajuste.gamma$estimate[2])
hist(x, main='Histograma', col='lightblue',ylab='Frecuencia',xlab='Demanda',
cex.main=0.9, cex.axis=0.8,prob=TRUE )
lines(axisx, valuesy, lwd = 2, col = "red")
boxplot(x, col='lightblue', main='Box plot',ylab='Demanda', cex.main=.9, cex.axis=.8)
plot(ecdf(x),main='Distibucion acumulada',col='lightblue',xlab='Demanda',ylab='Frecuencia',
cex.main=0.9, cex.axis=0.8,prob=TRUE)
z <- seq(min(x), max(x), (max(x)-min(x))/(length(x)-2))
lines(z,pgamma(z, ,rate=ajuste.gamma$estimate[2],shape=ajuste.gamma$estimate[1])
, lty=1,col='red',)
n <-length(x)
xq2  <- qgamma(ppoints(n), rate=ajuste.gamma$estimate[2], shape=ajuste.gamma$estimate[1])
Xsim  <- matrix(0, 100, n)
for(i in 1:100){
```

```
u2       <- rgamma(n,rate=ajuste.gamma$estimate[2],shape=ajuste.gamma$estimate[1])
Xsim[i,] <- u2
}
Xsim2      <- apply(Xsim, 1, sort)
d21        <- matrix(0, n, 1)
d22        <- matrix(0, n, 1)
for(i in 1:n){
d21[i]  <- quantile(Xsim2[i,], 0.025)
d22[i]  <- quantile(Xsim2[i,], 0.975)
}
d2med      <- apply(Xsim2, 1, median)
fy         <- range(sort(x), d21, d22)
plot(xq2,sort(x), xlab = quote("IG quantile"),ylab = "Empirical quantile",pch = 20,
ylim = fy)
par(new = T)
plot(xq2, d21, type = "l", ylim = fy, xlab = "", ylab = "")
par(new = T)
plot(xq2, d2med, type = "l", ylim = fy, xlab = "", ylab = "")
par(new = T)
plot(xq2, d22, type = "l", ylim = fy, xlab = "", ylab = "")
title(entitled)
Rsq <- cor(xq2, d2med)^2*100
print(paste0('Ajuste de envelopes:', Rsq))
}
#============= KS ==================
ks_gamma<-function(x){
ajuste.gamma<-fitdistr(x,"gamma")
g<-ks.test(x, "pgamma",ajuste.gamma$estimate[1],ajuste.gamma$estimate[2])$p.value
list(valor_p=g)
}
#======AIC============================
aic_gamma <- function(x) {
estimates <-fitdistr(x,"gamma")
alpha<-estimates$estimate[1]
beta<-estimates$estimate[2]
aic <-  -2 * log(prod(dgamma(x,alpha,beta ))) + 4
print(paste0(aic))
}
#============= BIC ==================
bic_gamma <- function(x){
ajuste.gamma <-fitdistr(x,"gamma")
alpha<-ajuste.gamma$estimate[1]
beta<-ajuste.gamma$estimate[2]
BIC    <- (-2) * log(prod(dgamma(x, alpha, beta))) + 2 * log(length(x))
print(paste0(BIC))
}
bicgamma(r.gamma)
#-----------------------------EOQ -------------------------------#
EOQ_gamma<-function(x,CL,CS,l)
{
ajuste.gamma<-fitdistr(x,"gamma")
E_gam<-ajuste.gamma$estimate[2]*ajuste.gamma$estimate[1]
```

```
result <- sqrt((2 *l*E_gam * CL)/CS)
print(paste0(result))
}
EOQ_gamma(r.gamma,3,0.1,3)
#----------------------------ROP -------------------------------#
ROP_gamma<-function(x,k, l)
{
ajuste.gamma<-fitdistr(x,"gamma")
E_gam<-ajuste.gamma$estimate[2]*ajuste.gamma$estimate[1]
V_gam<-ajuste.gamma$estimate[2]*(ajuste.gamma$estimate[1]^2)
xp<-qgamma(k,rate=ajuste.gamma$estimate[2],shape=l*ajuste.gamma$estimate[1])
kp<-(xp-l*E_gam)/(l*sqrt(V_gam))
result<- (l*E_gam)+ kp*l*sqrt(V_gam)
print(paste0(result))
}
#-----------------------articulos perecibles--------------------------------
q_gamma<-function(x,CP,CS,C)
{
ajuste.gamma<-fitdistr(x,"gamma")
RC<-(CP-C)/(CP+CS)
y <- qgamma(RC,rate=ajuste.gamma$estimate[2],shape=ajuste.gamma$estimate[1]
)
print(paste0(y))
}
q_gamma(r.gamma,40,0.1,3)
#------------Gausssiana-----------------------------------
#====== Histograma y Plots ======================
x.norm<-rnorm(n=200,m=10,sd=1)
ajuste.normal<-fitdistr(x,"normal")
graficos_norm<-function(x, entitled = "Envelopes")
{
par(mfrow=c(2,2))
minimum <- min(x) - sd(x)
maximum <- max(x) + sd(x)
axisx <- seq(minimum, maximum, by = (max(x)-min(x))/(length(x)-2))
ajuste.normal<-fitdistr(x,"normal")
valuesy <- dnorm(axisx, ajuste.normal$estimate[1],ajuste.normal$estimate[2])
hist(x, main='Histogarma', col='lightblue',ylab='Frecuencia',
cex.main=0.9, cex.axis=0.8,prob=TRUE )
lines(axisx, valuesy, lwd = 2, col = "red")
boxplot(x, col='lightblue', main='Box plot', cex.main=.9, cex.axis=.8)
plot(ecdf(x),main='Distibucion acumulada',col='lightblue',ylab='Frecuencia',
 cex.main=0.9, cex.axis=0.8,prob=TRUE)
z <- seq(min(x), max(x), (max(x)-min(x))/(length(x)-2))
lines(z,pnorm(z,m=ajuste.normal$estimate[1], sd = ajuste.normal$estimate[2]),
lty=1,col='red',)
n     <- length(x)
xq2  <- qnorm(ppoints(n), ajuste.normal$estimate[1], ajuste.normal$estimate[1])
Xsim  <- matrix(0, 100, n)
for(i in 1:100){
    u2       <- rnorm(n, m=ajuste.normal$estimate[1],sd= ajuste.normal$estimate[2])
    Xsim[i,] <- u2
```

```
}
Xsim2      <- apply(Xsim, 1, sort)
d21        <- matrix(0, n, 1)
d22        <- matrix(0, n, 1)
for(i in 1:n){
d21[i]  <- quantile(Xsim2[i,], 0.025)
d22[i]  <- quantile(Xsim2[i,], 0.975)
}
d2med      <- apply(Xsim2, 1, median)
fy         <- range(sort(x), d21, d22)
plot(xq2,sort(x), xlab = quote("IG quantile"),ylab = "Empirical quantile",pch = 20,
ylim = fy)
par(new = T)
plot(xq2, d21, type = "l", ylim = fy, xlab = "", ylab = "")
par(new = T)
plot(xq2, d2med, type = "l", ylim = fy, xlab = "", ylab = "")
par(new = T)
plot(xq2, d22, type = "l", ylim = fy, xlab = "", ylab = "")
title(entitled)
Rsq <- cor(xq2, d2med)^2*100
print(paste0('Ajuste de envelopes:', Rsq))
}
#------------bondad de Ajuste-----------------------------
#========= Shapiro-Wilk =============
shapirop<- function(x){
g<-shapiro.test(x)$p.value
list(valor_p=g)
}
shapirop(x.norm)
#=============== AIC ===================
aic_nor <- function(x, mu,sigma) {
norm<-fitdistr(x,"normal")
mu<-norm$estimate[1]
sigma<-norm$estimate[2]
aic <- -3* log(prod(dnorm(x, mu,sigma))) +4
print(paste0(aic))
}
#============= BIC ==================
bic_nor <- function(x){
norm<-fitdistr(x,"normal")
mu<-norm$estimate[1]
sigma<-norm$estimate[2]
BIC    <- (-2) * log(prod(dnorm(x, mu,sigma))) + 2 * log(length(x))
print(paste0(BIC))
}
#===========================================================
ROP_nor<-function(x,k,l)
{
ajuste.normal<-fitdistr(x,"normal")
mu<-ajuste.normal$estimate[1]
sigma<-ajuste.normal$estimate[2]
xp        <- qnorm(k, mu, l*sigma)
```

```
kp    <- (xp-(l*mu))/(l*sigma)
result    <- (l*mu)+ (sqrt(l)*sigma)
#list(ROP=result)
print(paste0('ROP: ', result))
}
#-----------------------articulos perecibles--------------------------------
q_norm<-function(x,CP,CS,C)
{
ajuste.normal<-fitdistr(x,"normal")
mu<-ajuste.normal$estimate[1]
sigma<-ajuste.normal$estimate[2]
RC<-(CP-C)/(CP+CS)
F_Z<-qnorm(RC,mu,sigma)
print(paste0(F_Z))
}
#-----------------------------EOQ sin escasez -------------------------------#
EOQ_norm<-function(x,CL,CS,l)
{
ajuste.normal<-fitdistr(x,"normal")
mu<-ajuste.normal$estimate[1]
result <- sqrt((2 *l*mu * CL)/CS)
print(paste0(result))
}
#------------Inverse Gaussian------------------------------
graficos_ig <- function(x, entitled = "Data set"){
par(mfrow=c(2,2))
minimum <- min(x) - sd(x)
maximum <- max(x) + sd(x)
axisx <- seq(minimum, maximum, by = 0.1)
w<-mleig(x)
valuesy <- digt(axisx, mu = w$muEstimate, lambda = w$lambdaEstimate,
nu = 1.0, kernel = "normal",log = FALSE)
hist(x, main='Histograma', col='lightblue',ylab='Frecuencia',xlab='Demanda',
cex.main=0.9, cex.axis=0.8,prob=TRUE )
lines(axisx, valuesy, lwd = 2, col = "red")
boxplot(x, col='lightblue', main='Box plot',ylab='Demanda', cex.main=.9, cex.axis=.8)
plot(ecdf(x),main='Distibucion acumulada',,xlab='Demanda',col='lightblue',
ylab='Frecuencia',
cex.main=0.9, cex.axis=0.8,prob=TRUE)
z <- seq(min(x), max(x), (max(x)-min(x))/(length(x)-2))
lines(z,pigt(z,mu=w$muEstimate, lambda= w$lambdaEstimate),lty=1,col='red')
n     <- length(x)
xq2  <- qigt(ppoints(n), mu =w$muEstimate, lambda = w$lambdaEstimate)
Xsim  <- matrix(0, 100, n)
for(i in 1:100){
u2       <- rigt(n, mu = w$muEstimate, lambda = w$lambdaEstimate)
Xsim[i,] <- u2
}
Xsim2      <- apply(Xsim, 1, sort)
d21        <- matrix(0, n, 1)
d22        <- matrix(0, n, 1)
for(i in 1:n){
```

```
d21[i]  <- quantile(Xsim2[i,], 0.025)
d22[i]  <- quantile(Xsim2[i,], 0.975)
}
d2med      <- apply(Xsim2, 1, median)
fy         <- range(sort(x), d21, d22)
plot(xq2,sort(x), xlab = quote("IG quantile"),ylab = "Empirical quantile",pch = 20,
ylim = fy)
par(new = T)
plot(xq2, d21, type = "l", ylim = fy, xlab = "", ylab = "")
par(new = T)
plot(xq2, d2med, type = "l", ylim = fy, xlab = "", ylab = "")
par(new = T)
plot(xq2, d22, type = "l", ylim = fy, xlab = "", ylab = "")
title(entitled)
Rsq <- cor(xq2, d2med)^2*100
print(paste0('Ajuste de envelopes:', Rsq))
}
#============KS=========================
ks_ig<-function(x){
mleig(x)
alpha_ig<-mleig(x)$muEstimate
beta_ig<-mleig(x)$lambdaEstimate
kswei<- ks.test(x, "pigt", mu = alpha_ig, lambda =beta_ig)$p.value
print(paste0(kswei))
}
#======AIC============================
aic_ig <- function(x) {
ig<-mleig(x)
alpha_ig<-ig$muEstimate
beta_ig<-ig$lambdaEstimate
aicig <-  -2 * log(prod(digt(x, mu = alpha_ig, lambda = beta_ig, nu = 1.0,
kernel = "normal",log = FALSE))) + 4
print(paste0(aicig))
}
#============= BIC ==================
bic_ig <- function(x){
ig<-mleig(x)
alpha_ig<-ig$muEstimate
beta_ig<-ig$lambdaEstimate
BIC    <- (-2) * log(prod(digt(x, mu = alpha_ig, lambda = beta_ig, nu = 1.0,
kernel = "normal",log = FALSE))) + 2 * log(length(x))
print(paste0(BIC))
}
#-----------------------------EOQ  -------------------------------#
EOQ_ig<-function(x,CL,CS,l)
{
estimates <-mleig(x)
E_igaus<-mleig(x)$muEstimate
result <- sqrt((2 *l*E_igaus * CL)/CS)
print(paste0(result))
}
#-----------------------------ROP -------------------------------#
```

```
ROP_ig<-function(x,k, l)
{
estimates <-mleig(x)
E_ig<-estimates$muEstimate
V_ig<-estimates$muEstimate^3/estimates$lambdaEstimate
xp<-qigt(k, mu = estimates$muEstimate, lambda =l*estimates$lambdaEstimate,
nu = 1.0, kernel = "normal")
kp<- (xp-l*E_ig)/(l*sqrt(V_ig))
result    <- (l*E_ig)+ kp*l*sqrt(V_ig)
print(paste0(result))
}
#----------------------articulos perecibles--------------------------------
q_ig<-function(x,CP,CS,C)
{
estimates <-mleig(x)
RC<-(CP-C)/(CP+CS)
y <- qigt(RC ,mu=estimates$muEstimate, lambda=estimates$lambdaEstimate,
nu = 1.0, kernel = "normal")
print(paste0(y))
}
#============================================================
#------------Lognormal---------------------------------------
#rln<-rlnorm(200, 5, 30)
rln<-rlnorm(200, 0.1, 0.3)
graficos_rln<-function(x, entitled = "Data set")
{
par(mfrow=c(2,2))
minimum <- min(x) - sd(x)
maximum <- max(x) + sd(x)
axisx <- seq(minimum, maximum, by = 0.1)
ln<-fitdistr(x, "lognormal")
meanlog = ln$estimate[1]
sdlog = ln$estimate[2]
valuesy <- dlnorm(axisx, meanlog,sdlog)
hist(x, main='Histograma', col='lightblue',ylab='Frecuencia',xlab='Demanda',
cex.main=0.9, cex.axis=0.8,prob=TRUE )
lines(axisx, valuesy, lwd = 2, col = "red")
boxplot(x, col='lightblue', main='Box plot',ylab='Demanda', cex.main=.9, cex.axis=.8)
plot(ecdf(x),main='Distibucion acumulada',col='lightblue',xlab='Demanda',
ylab='Frecuencia',cex.main=0.9, cex.axis=0.8,prob=TRUE)
z <- seq(min(x), max(x), (max(x)-min(x))/(length(x)-2))
lines(z,plnorm(z,meanlog, sdlog),lty=1,col='red')
n <-length(x)
xq2  <- qlnorm(ppoints(n),meanlog = meanlog,sdlog =sdlog)
Xsim  <- matrix(0, 100, n)
for(i in 1:100){
u2       <- rlnorm(length(x), meanlog, sdlog)
Xsim[i,] <- u2
}
Xsim2      <- apply(Xsim, 1, sort)
d21        <- matrix(0, n, 1)
d22        <- matrix(0, n, 1)
```

```
for(i in 1:n){
d21[i]  <- quantile(Xsim2[i,], 0.025)
d22[i]  <- quantile(Xsim2[i,], 0.975)
}
d2med      <- apply(Xsim2, 1, median)
fy         <- range(sort(x), d21, d22)
plot(xq2,sort(x), xlab = quote("IG quantile"),ylab = "Empirical quantile",pch = 20,
ylim = fy)
par(new = T)
plot(xq2, d21, type = "l", ylim = fy, xlab = "", ylab = "")
par(new = T)
plot(xq2, d2med, type = "l", ylim = fy, xlab = "", ylab = "")
par(new = T)
plot(xq2, d22, type = "l", ylim = fy, xlab = "", ylab = "")
title(entitled)
Rsq <- cor(xq2, d2med)^2*100
print(paste0('Ajuste de envelopes:', Rsq))
}
#===========KS=================
kslogn<-function(x){
mleln<-fitdistr(x, "lognormal")
ks<- ks.test(x, "plnorm", mean(log(x)), sd(log(x)))$p.value
print(paste0(ks))
}
#======AIC============================
aic_ln <- function(x) {
mleln<-fitdistr(x, "lognormal")
meanlog = mleln$estimate["meanlog"]
sdlog = mleln$estimate["sdlog"]
aic <- -2 * log(prod(dlnorm(x, meanlog,sdlog))) +4
print(paste0(aic))
}
#============= BIC ==================
bicln <- function(x){
mleln<-fitdistr(x, "lognormal")
meanlog = mleln$estimate["meanlog"]
sdlog = mleln$estimate["sdlog"]
BIC    <- (-2) * log(prod(dlnorm(x, meanlog,sdlog))) + 2 * log(length(x))
print(paste0(BIC))
}
#-----------------------------EOQ -------------------------------#
EOQ_ln<-function(x,CL,CS,l)
{
mle_ln<-fitdistr(x, "lognormal")
meanlog = mle_ln$estimate[1]
result <- sqrt((2 *l*meanlog* CL)/CS)
print(paste0(result))
}
#-----------------------------ROP-------------------------------#
ROPln<-function(x,k, l)
{
mle_ln<-fitdistr(x, "lognormal")
```

```
meanlog = mle_ln$estimate[1]
sdlog = mle_ln$estimate[2]
xp<- qlnorm(k,meanlog = meanlog,sdlog =l*sdlog)
kp<-(xp-l*meanlog)/(l*sdlog)
result<-(l*meanlog)+ kp*l*sdlog
print(paste0(result))
}
#----------------------articulos perecibles--------------------------------
q_ln<-function(x,CP,CS,C)
{
mle_ln<-fitdistr(x, "lognormal")
meanlog = mle_ln$estimate[1]
sdlog = mle_ln$estimate[2]
RC<-(CP-C)/(CP+CS)
y <- qlnorm(RC,meanlog = meanlog,sdlog =sdlog)
print(paste0(y))
}
#------------Weibull------------------------------------------
graficos_weibull<-function(x, entitled = "Envelopes")
{
par(mfrow=c(2,2))
minimum <- min(x) - sd(x)
maximum <- max(x) + sd(x)
axisx <- seq(minimum, maximum, by = 0.1)
ajuste.wei<-fitdistr(x, densfun="weibull")
valuesy <- dweibull(axisx, shape =ajuste.wei$estimate[1],
scale = ajuste.wei$estimate[2])
hist(x, main='Histograma', col='lightblue',ylab='Frecuencia',xlab='Demanda',
cex.main=0.9, cex.axis=0.8,prob=TRUE )
lines(axisx, valuesy, lwd = 2, col = "red")
boxplot(x, col='lightblue', main='Box plot',ylab='Demanda',  cex.main=.9, cex.axis=.8)
plot(ecdf(x),main='Distibucion acumulada',col='lightblue',xlab='Demanda' ,
ylab='Frecuencia',cex.main=0.9, cex.axis=0.8,prob=TRUE)
z <- seq(min(x), max(x), (max(x)-min(x))/(length(x)-2))
lines(z,pweibull(z ,scale=ajuste.wei$estimate[2], shape=ajuste.wei$estimate[1]),
lty=1,col='red',)
n <-length(x)
xq2  <- qweibull(ppoints(n),scale=ajuste.wei$estimate[2], shape=ajuste.wei$estimate[1])
Xsim  <- matrix(0, 100, n)
for(i in 1:100){
u2       <- rweibull(length(x),scale=ajuste.wei$estimate[2],
shape=ajuste.wei$estimate[1])
Xsim[i,] <- u2
}
Xsim2      <- apply(Xsim, 1, sort)
d21        <- matrix(0, n, 1)
d22        <- matrix(0, n, 1)
for(i in 1:n){
d21[i]  <- quantile(Xsim2[i,], 0.025)
d22[i]  <- quantile(Xsim2[i,], 0.975)
}
d2med      <- apply(Xsim2, 1, median)
```

```
fy          <- range(sort(x), d21, d22)
plot(xq2,sort(x), xlab = quote("IG quantile"),ylab = "Empirical quantile",
pch = 20, ylim = fy)
par(new = T)
plot(xq2, d21, type = "l", ylim = fy, xlab = "", ylab = "")
par(new = T)
plot(xq2, d2med, type = "l", ylim = fy, xlab = "", ylab = "")
par(new = T)
plot(xq2, d22, type = "l", ylim = fy, xlab = "", ylab = "")
title(entitled)
Rsq <- cor(xq2, d2med)^2*100
print(paste0('Ajuste de envelopes:', Rsq))
}
#============KS=====================
ks_wei <- function(x){
ajuste.wei<-fitdistr(x, densfun="weibull")
w<-ks.test(x,"pweibull", shape=ajuste.wei$estimate[1],scale=ajuste.wei$estimate[2])$p.value
print(paste0(w))
}
#======AIC============================
aic_wei <- function(x,alpha,beta) {
ajuste.weibull<-fitdistr(x, densfun="weibull")
aic <- -2 * log(prod(dweibull(x = x, shape =ajuste.weibull$estimate[1],
scale = ajuste.weibull$estimate[2]))) + 4
print(paste0(aic))
}
#============= BIC ==================
bic_wei <- function(x){
ajuste.weibull<-fitdistr(x, densfun="weibull")
BIC    <- (-2) * log(prod(dweibull(x = x, shape =ajuste.weibull$estimate[1],
scale =ajuste.weibull$estimate[2]))) + 2 * log(length(x))
print(paste0(BIC))
}
#-----------------------------EOQ -------------------------------#
EOQwei<-function(x,CP,CS,l)
{
ajuste.weibull<-fitdistr(x, densfun="weibull")
E_wei<-ajuste.weibull$estimate[2]* factorial(1/ajuste.weibull$estimate[1] +1)
result <- sqrt((2 *l*E_wei * CP)/CS)
print(paste0(result))
}
#-----------------------------ROP -------------------------------#
ROP_wei<-function(x,k, l)
{
ajuste.weibull<-fitdistr(x, densfun="weibull")
scale<-ajuste.weibull$estimate[2]
shape<-ajuste.weibull$estimate[1]
E_wei<-scale* factorial(1/shape +1)
V_wei<-scale* ((factorial(2/shape+1))-(factorial(1/shape+1))^2)
xp<- qweibull(k,shape,l*scale)
kp<-(xp-(l*E_wei))/(l*sqrt(V_wei))
result    <- l*E_wei+ kp*l*sqrt(V_wei)
```

```
print(paste0('ROP:',result))
}
#----------------------articulos perecibles--------------------------------
q_weibull<-function(x,CP,CS,C)
{
ajuste.weibull<-fitdistr(x, densfun="weibull")
RC<-(CP-C)/(CP+CS)
y <- qweibull(RC,shape,scale)
print(paste0(y))
}
#################################################################################
#Algoritmo DE####################################################################
#################################################################################
require("DEoptim")
require("compiler")
DEMIDA<-function(hol,CTR,Cip,xis,sd,lt,slt)
{
imax<-length(xis)
sdlt<-sqrt(xis^2*slt^2+sd^2*lt)
CV_dlt<-sdlt/(xis*lt)
alpha<-1/CV_dlt^2
Beta<-sdlt^2/(xis*lt)
alpha.BS<- matrix(0, nr = imax, nc = 1)
Beta.BS<-matrix(0, nr = imax, nc = 1)
sdlt<-sqrt(xis^2*slt^2+sd^2*lt)
Lot.Size.Gamma<-matrix(0, nr = imax, nc = 1)
Lot.Size.Norm<-matrix(0, nr = imax, nc = 1)
Lot.Size.BS<-matrix(0, nr = imax, nc = 1)
Safety.factor.Gamma<-matrix(0, nr = imax, nc = 1)
Safety.factor.Norm<-matrix(0, nr = imax, nc = 1)
Safety.factor.BS<-matrix(0, nr = imax, nc = 1)
Minimal.Total.Cost.Gamma<-matrix(0, nr = imax, nc = 1)
Minimal.Total.Cost.Norm<-matrix(0, nr = imax, nc = 1)
Minimal.Total.Cost.BS<-matrix(0, nr = imax, nc = 1)
Shortage.Gamma<-matrix(0, nr = imax, nc = 1)
Shortage.Norm<-matrix(0, nr = imax, nc = 1)
Shortage.BS<- matrix(0, nr = imax, nc = 1)

# GENERATING RANDOM PARAMETERS BASED ON LITERATURE REVIEW
for (i in 1:imax) {
# Determining Alpha and Beta BS
 AlphaBS<- function(x) {
  a<-x[1]
  return(CV_dlt[i]-a*sqrt(4+5*a^2)/(2+a^2))
 }
 aux<- uniroot(AlphaBS,c(0,100000))$root
 alpha.BS[i]<-aux

 BetaBS<- function(x) {
  b<-x[1]
  #return(xis[i]*lt[i]-b*(1+alpha.BS[i]^2/2))
  return(sdlt[i]^2- alpha.BS[i]^2*b^2*(1+5* alpha.BS[i]^2/4))
```

```
 }
 aux2<-uniroot(BetaBS,c(0,100000))$root
 Beta.BS[i]<-aux2
}

## Modelling Holding Cost
for (i in 1:imax) {

 HC <- function(x) {
  Q<-x[1]
  k<-x[2]
  Total.HC <- hol[i]*(Q/2+sdlt[i]*k)
  return(Total.HC)
 }
## Modelling Stock-out Cost
 SC.Gamma<- function(x) {
  Q<-x[1]
  k<-x[2]
  Shortage.Gamma[i]<-(1-pgamma(xis[i]*lt[i]+sdlt[i]*k, shape=alpha[i], scale = Beta[i]))
  Total.SC.Gamma <-Cip[i]*xis[i]*12/Q*Shortage.Gamma[i]
  return(Total.SC.Gamma)
 }
 SC.Norm<- function(x) {
  Q<-x[1]
  k<-x[2]
  Shortage.Norm[i]<-(1-pnorm(k,0,1))
  Total.SC.Norm <-Cip[i]*xis[i]*12/Q*Shortage.Norm[i]
  return(Total.SC.Norm)
 }
 SC.BS<- function(x) {
  Q<-x[1]
  k<-x[2]
  Shortage.BS[i]<- tryCatch(1 - pgbs(xis[i]*lt[i]+(k)*sdlt[i], alpha.BS[i],
  Beta.BS[i]), error = function(e) NaN)
  Total.SC.BS <-Cip[i]*xis[i]*12/Q*Shortage.BS[i]
  return(Total.SC.BS)
 }
## Modelling Ordering Costs
 OC<- function(x) {
  Q<-x[1]
  k<-x[2]
  Total.OC<-xis[i]*12*CTR[i]/Q+0*k
  return(Total.OC)
 }
##Building Objective Function
 TC.Gamma<- function(x) {
  Q<-x[1]
  k<-x[2]
  Total.Cost.Gamma<- HC(x)+ OC(x) +SC.Gamma(x)
  return(Total.Cost.Gamma)
 }
 TC.Norm<- function(x) {
```

```
  Q<-x[1]
  k<-x[2]
  Total.Cost.Norm<- HC(x)+ OC(x) +SC.Norm(x)
  return(Total.Cost.Norm)
 }
 TC.BS<- function(x) {
  Q<-x[1]
  k<-x[2]
  Total.Cost.BS<- HC(x)+ OC(x) +SC.BS(x)
  return(Total.Cost.BS)
 }
## Optimizing Scenarios
 A<-DEoptim(fn =TC.Gamma, lower = c(0,0), upper =c(max(round(xis[i]*36,0),1), 10),
 control = list(NP = 100, itermax = 100, trace = FALSE))
 B<-DEoptim(fn =TC.Norm, lower = c(0,0), upper =c(max(round(xis[i]*36,0),1), 10),
 control = list(NP = 100, itermax = 100, trace = FALSE))
 C<- tryCatch(DEoptim(fn =TC.BS, lower = c(0,0), upper =c(max(round(xis[i]*36,0),1), 1000),
  control = list(NP = 100, itermax = 100, trace = FALSE)), error = function(e) NaN)

## Saving Results

 Lot.Size.Gamma[i]<- round(A$optim$bestmem[1],0)
 Safety.factor.Gamma[i]<- round(A$optim$bestmem[2],2)
 Minimal.Total.Cost.Gamma[i]<- A$optim$bestval
 Lot.Size.Norm[i]<- round(B$optim$bestmem[1],0)
 Safety.factor.Norm[i]<- round(B$optim$bestmem[2],2)
 Minimal.Total.Cost.Norm[i]<- B$optim$bestval
 if (length(C) != 1) {
  Lot.Size.BS[i]<- round(C$optim$bestmem[1],0)
  Safety.factor.BS[i]<- round(C$optim$bestmem[2],2)
  Minimal.Total.Cost.BS[i]<- C$optim$bestval
 } else {
  Lot.Size.BS[i]<- NA
  Safety.factor.BS[i]<- NA
  Minimal.Total.Cost.BS[i]<- NA
}
 print(cbind(i,Lot.Size.Gamma[i],Safety.factor.Gamma[i],Minimal.Total.Cost.Gamma[i]
 ,Lot.Size.Norm[i],Safety.factor.Norm[i],Minimal.Total.Cost.Norm[i] ,Lot.Size.BS[i],
 Safety.factor.BS[i],Minimal.Total.Cost.BS[i]))
  flush.console()
}
RES<-data.frame(i, xis, sd, lt, slt, CTR, hol, sdlt, Cip, alpha, Beta, CV_dlt,
alpha.BS, Beta.BS,Lot.Size.Gamma,Safety.factor.Gamma,Minimal.Total.Cost.Gamma,
Shortage.Gamma, Lot.Size.Norm,Safety.factor.Norm,Minimal.Total.Cost.Norm, Shortage.Norm,
 Lot.Size.BS,Safety.factor.BS,Minimal.Total.Cost.BS, Shortage.BS)
write.table(RES,sep=",","RESULTS NUMERICAL.csv")
## Code Ends
}
```

Tables with results and ID products

Table 3.8: summary of statistical and inventory models for the indicated product.

ID	Inventory model	Statistical distribution	λ (unit/day)	σ (unit/day)	$k_{0.95}$	SS (unit/day)	Q (unit)	CR	d^0 (unit)	r (unit)
P1	P	BS-t	178.920	152.8749	-	-	-	0.8968	364.0089	-
P2	P	BS-t	20.3812	13.3375	-	-	-	0.8968	31.7474	-
P3	P	BS-t	13.2742	2.34147	-	-	-	0.8968	15.7216	-
P4	P	Weibull	17.4480	5.85620	-	-	-	0.8968	24.9878	-
P5	EOQ	Constant	4.0000	0.0000	-	-	1253.037	-	-	12.0000
P6	P	Average	32.1667	13.1821	-	-	-	0.8968	48.8193	-
P7	P	Normal	6.1370	3.1150	-	-	-	0.8968	10.0721	-
P8	EOQ	Average	17.0625	19.1188	1.645	94.34	2988.302	-	-	82.6352
P9	EOQ	Constant	1.0000	0.0000	-	-	723.441	-	-	3.0000
P10	P	Weibull	21.8918	10.0220	-	-	-	0.8968	35.1937	-
P11	P	Gamma	71.5276	98.5146	-	-	-	0.8968	187.8697	-
P12	EOQ	Constant	1.0000	0.0000	-	-	723.441	-	-	3.0000
P13	JIT	Weibull	23.2934	5.0932	-	-	-	-	-	-
P14	P	Normal	36.0351	23.6604	-	-	-	0.5008	36.0826	-
P15	P	Weibull	14.0187	4.5630	-	-	-	0.8968	19.8742	-
P16	P	Average	3.5790	1.5342	-	-	-	0.8968	5.51704	-
P17	P	BS-t	18.9254	16.0642	-	-	-	0.8968	33.3047	-
P18	P	BS-t	11.5344	7.8153	-	-	-	0.8968	18.1837	-
P19	P	Weibull	54.5543	35.4026	-	-	-	0.8968	102.2442	-
P20	P	Average	4.2733	2.2908	-	-	-	0.5008	4.27793	-
P21	P	Weibull	8.4660	2.9954	-	-	-	0.5008	8.41548	-
P22	P	Weibull	15.5641	5.6989	-	-	-	0.8968	22.9691	-
P23	EOQ	Average	2.1308	0.7820	1.645	3.86	1056.018	-	-	7.6786
P24	EOQ	Average	3.8333	1.2673	1.645	6.25	1416.418	-	-	13.5845
P25	EOQ	Average	3.5926	1.0099	1.645	4.98	1371.220	-	-	12.4390
P26	EOQ	Average	0.9130	0.5771	1.645	2.85	691.272	-	-	3.68832
P27	EOQ	Constant	4.0000	0.0000	-	-	1253.030	-	-	12.0000
P28	P	Gamma	41.3161	16.9635	-	-	-ï¿½	0.8968	63.5827	-
P29	EOQ	Constant	4.0000	0.0000	-	-	1253.030	-	-	12.0000
P30	EOQ	Constant	4.0000	0.0000	-	-	1253.030	-	-	12.0000
P31	EOQ	Constant	1.5000	0.0000	-	-	886.031	-	-	4.5000
P32	EOQ	Constant	1.0000	0.0000	-	-	723.441	-	-	3.0000
P33	P	BS-t	92.0679	62.2745	-	-	-	0.8968	159.6481	-
P34	EOQ	Gamma	6.7358	4.0623	1.818	12.79	1877.584	-	-	32.9983
P35	JIT	Constant	1.0000	0.0000	-	-	-	-	-	-
P36	P	BS-t	11.5092	1.0275	-	-	-	0.8968	17.7578	-
P37	P	BS-t	8.7952	5.6716	-	-	-	0.8968	15.8991	-
P38	P	Average	15.8448	7.5156	-	-	-	0.8968	25.3391	-
P39	P	Average	65.8700	47.3483	-	-	-	0.5008	65.9650	-
P40	P	Weibull	17.3387	5.1059	-	-	-	0.8968	23.8207	-
P41	EOQ	Average	2.8677	1.5826	1.645	7.81	1225.084	-	-	11.2061
P42	P	BS-t	23.1870	35.5482	-	-	-	0.5008	15.4503	-
P43	EOQ	Gamma	2.3333	1.8067	1.857	5.81	1.105.074	-	-	128.126
P44	JIT	BS	80.2909	13.1822	-	-	-	-	-	-
P45	EOQ	Average	1.4940	0.9832	1.645	4.85	884.250	-	-	6.0991
P46	EOQ	Gamma	11.0058	4.3185	1.763	13.19	2400.009	-	-	46.2062
P47	EOQ	Average	32.5714	3.5523	1.645	17.53	4128.778	-	-	103.5573
P48	P	Weibull	45.7552	17.2627	-	-	-	0.8968	68.2554	-
P49	P	Average	8.0455	2.9355	-	-	-	0.8968	11.7538	-
P50	P	Average	2.0750	1.4167	-	-	-	0.8968	3.8647	-
P51	P	Average	10.2593	4.2661	-	-	-	0.8968	15.6485	-
P52	EOQ	Constant	10.000	0.0000	-	-	ï¿½723.441	-	-	3.0000
P53	EOQ	Average	3.61225	0.8371	1.645	4.13	1374.965	-	-	12.2136
P54	EOQ	Constant	4.0000	0.0000	-	-	1253.037	-	-	12.0000
P55	EOQ	Constant	4.0000	0.0000	-	-	1253.037	-	-	12.0000
P56	EOQ	Gamma	2.2026	1.5400	1.841	4.91	1073.667	-	-	11.5175
P57	P	Average	5.3250	0.5516	-	-ï¿½	-	0.8968	6.02185	-
P58	EOQ	Average	42.3333	13.5154	1.645	66.69	4707.002	-	-	149.2309
P59	EOQ	Average	4.1515	0.7954	1.645	3.92	ï¿½1474.030	-	-	13.7628
P60	EOQ	Constant	4.0000	0.0000	-	-	1253.037	-	-	12.0000
P61	EOQ	Gamma	8.6085	5.3069	1.821	16.74	2122.589	-	-	42.5649
P62	P	BS-t	19.8527	5.8261	-	-	-	0.8968	29.4351	-
P63	P	Average	11.0061	3.0065	-	-	-	0.8968	14.8472	-
P64	P	BS-t	25.2673	13.2933	-	-	-	0.8968	37.5170	-
P65	P	Weibull	14.9037	5.4825	-	-	-	0.8968	22.0310	-
P66	P	Average	10.7543	2.7744	-	-	-	0.8968	14.2932	-
P67	P	BS-t	14.8657	2.7844	-	-	-	0.8968	22.3732	-
P68	P	BS-t	53.5429	64.6308	-	-	-	0.8968	127.0268	-
P69	P	Weibull	83.4886	66.2302	-	-	-	0.8968	171.6083	-
P70	EOQ	Gamma	3.1538	2.2651	1.845	7.24	1284.764	-	-	16.7000
P71	P	Constant	3.0000	0.0000	-	-	-	0.8968	65.5800	-

Table 3.9: (Table 3.8 continued) summary of statistical and inventory models for the indicated product.

ID	Inventory model	Statistical distribution	λ (unit/day)	σ (unit/day)	$k_{0.95}$	SS (unit/day)	Q (unit)	CR	d^0 (unit)	r (unit)
P72	P	Weibull	22.7972	13.0891	-	-	-	0.8968	40.4072	-
P73	EOQ	Gamma	25.6761	27.1210	1.913	89.86	3665.788	-	-	166.8877
P74	EOQ	Constant	1.0000	0.0000	-	-	ï¿$\frac{1}{2}$723.441	-	-	30000
P75	EOQ	Average	2.3501	0.7617	1.645	3.76	1106.560	-	-	8.2717
P76	EOQ	Constant	1.0000	0.0000	-	-	ï¿$\frac{1}{2}$723.441	-	-	3.0000
P77	JIT	Average	1.6931	0.4624	-	-	-	-	-	-
P78	P	Average	25.0652	14.7219	-	-	-	0.5008	25.0947	-
P79	P	Average	16.8333	1.1691	-	-	-	0.8968	18.3102	-
P80	EOQ	Average	34.3846	19.7718	1.645	97.56	4242.142	-	-	135.6755
P81	EOQ	Average	37.3333	18.0870	1.645	89.25	4420.298	-	-	141.7510
P82	P	BS-t	135.9223	142.046	-	-	-	0.8968	308.8345	-
P83	EOQ	Average	296.81	0.9211	1.645	4.55	1.246.354	-	-	10.4193
P84	P	Gamma	34.6047	15.0383	-	-	-	0.8968	54.3573	-
P85	EOQ	Constant	1.0000	0.0000	-	-	723.441	-	-	3.0000
P86	P	Weibull	9.7969	4.7464	-	-	-	0.8968	16.1245	-
P87	P	Normal	23.4211	7.9465	-	-	-	0.8968	33.4597	-
P88	EOQ	Average	2.9364	1.1214	1.645	5.53	12.39687	-	-	10.6538
P89	P	Weibull	4.8498	2.4073	-	-	-	0.8968	8.0645	-

Table 3.10: values of $SQ_{42,j}$ (in kg) and $CM_{42,j}$, $VCM_{42,j}$, $OC_{42,j}$ and $SC_{42,j}$ (in US$) for the indicated system in the product P42 and jth week, with $j = 1, \ldots, 27$.

Non-optimized system													
$SQ_{42,j}$													
2.00	3.25	12.5	2.90	0.95	1.35	0.90	2.90	0.65	2.40	1.60	0.95	2.65	1.80
0.48	0.95	1.43	0.95	2.10	4.05	0.95	1.90	5.70	3.55	6.75	5.90	6.60	
$VCM_{42,j}$													
-188.1	-268.9	-136.4	-232.1	-359.0	-34.9	-104.4	-317.5	8.6	-249.2	-96.87	-169.2	-250.2	-168.4
13.84	-204.5	-285.8	-420.7	-183.0	-441.8	11.17	-144.0	-574.8	-240.9	-228.9	-114.5	-166.2	
$OC_{42,j}$													
1.76	1.76	0.88	1.76	2.64	0.88	0.88	1.76	0.88	0.88	1.76	0.88	1.76	0.88
0.88	1.76	2.64	1.76	1.76	1.76	1.76	1.76	2.64	2.64	1.76	0.88	0.88	
$SC_{42,j}$													
0.05	0.09	0.33	0.08	0.03	0.04	0.02	0.08	0.02	0.06	0.04	0.03	0.07	0.05
0.01	0.03	0.04	0.03	0.06	0.11	0.03	0.05	0.15	0.09	0.18	0.16	0.17	
$CM_{42,j}$													
-189.9	-270.73	-137.6	-233.9	-37.61	-105.3	-318.4	6.79	-250.1	-97.81	-171.0	-251.1	-170.2	12.91
-205.4	-184.8	-444.4	9.38	-145.8	-576.7	-242.7	-230.7	-117.3	-169.0	-361.0	-286.8	-421.7	
Optimized system													
$SQ_{42,j}$													
-6.55	1.88	5.18	-7.82	3.73	1.45	-1.55	10.90	-4.77	9.13	-2.05	2.18	2.95	13.40
3.45	10.08	9.90	16.85	-0.32	-10.87	16.30	-1.82	13.13	9.90	5.98	10.95	-0.87	
$VCM_{42,j}$													
-76.02	-311.3	-169.7	-272.2	-89.47	-42.08	-111.5	-202.2	14.03	-349.1	-48.06	-189.6	-107.8	-198.2
-89.36	-150.8	-273.7	-292.3	-143.3	-453.6	-361.6	15.54	-239.5	-299.5	-298.4	-245.4	-171.7	
$SC_{42,j}$													
-0.13	0.04	0.10	-0.16	0.08	0.03	-0.03	0.22	-0.10	0.18	-0.04	0.04	0.06	0.27
0.07	0.20	0.20	0.34	-0.01	-0.22	0.33	-0.04	0.27	0.20	0.12	0.22	-0.02	
$OC_{42,j}$													
1.76	4.40	2.64	2.64	4.40	0.88	0.88	1.76	3.52	0.88	4.40	0.88	2.64	1.76
3.52	1.76	6.16	4.40	1.76	3.52	5.28	2.64	4.40	6.16	2.64	4.40	3.52	
$CM_{42,j}$													
-77.65	-315.7	-172.5	-274.7	-93.95	-42.99	-112.4	-204.2	10.61	-350.1	-52.42	-190.5	-110.5	-200.3
-92.95	-152.7	-280.1	-297.0	-145.1	-456.9	-367.3	12.94	-244.1	-305.9	-301.2	-250.1	-175.2	

Table 3.11: differential of optimized values and CP of VCM, OC, SC and CM for the indicated ID.

ID	CM (US$)	CP (%)	ID	VCM (US$)	CP (%)	ID	SC (US$)	CP (%)	ID	OC (US$)	CP (%)
P33	3276.90	17.51%	P33	966.11	11.35%	P33	2426.91	21.14%	P67	62.71	18.27%
P48	2500.78	30.87%	P48	799.17	20.74%	P48	1799.26	36.81%	P34	11.44	21.61%
P69	2010.00	41.62%	P14	685.69	28.79%	P69	1420.40	49.18%	P45	11.44	24.94%
P28	1624.89	50.30%	P69	675.82	36.73%	P28	1380.08	61.20%	P56	11.44	28.27%
P84	1593.27	58.81%	P64	634.39	44.18%	P84	1132.87	71.07%	P88	11.44	31.61%
P17	1073.65	64.55%	P84	558.94	50.75%	P17	873.65	78.68%	P46	10.56	34.68%
P64	844.19	69.06%	P42	389.60	55.33%	P4	473.88	82.81%	P41	10.56	37.76%
P14	707.36	72.84%	P28	326.63	59.16%	P62	312.35	85.53%	P61	10.50	40.82%
P4	631.81	76.22%	P17	302.93	62.72%	P64	246.75	87.68%	P73	10.42	43.86%
P11	438.88	78.56%	P18	272.96	65.93%	P10	243.32	89.80%	P27	9.68	46.68%
P62	379.13	80.59%	P4	254.52	68.92%	P11	240.01	91.89%	P29	9.68	49.50%
P18	373.97	82.59%	P11	254.29	71.91%	P2	143.32	93.14%	P59	8.80	52.06%
P42	371.13	84.57%	P10	179.76	74.02%	P19	138.05	94.34%	P76	8.80	54.63%
P10	350.00	86.44%	P40	165.24	75.96%	P18	132.59	95.49%	P60	8.69	57.16%
P67	287.02	87.97%	P78	132.46	77.52%	P67	100.43	96.37%	P83	7.92	59.46%
P19	219.19	89.15%	P62	126.61	79.00%	P40	78.16	97.05%	P9	7.92	61.77%
P40	218.76	90.31%	P67	123.89	80.46%	P72	50.24	97.49%	P55	7.92	64.08%
P72	158.21	91.16%	P72	116.76	81.83%	P22	39.13	97.83%	P47	7.92	66.39%
P2	154.77	91.99%	P37	103.82	83.05%	P16	35.45	98.14%	P80	7.92	68.69%
P78	145.96	92.77%	P22	103.04	84.26%	P15	29.73	98.40%	P75	7.92	71.00%
P22	117.54	93.40%	P51	93.40	85.36%	P14	29.59	98.65%	P32	7.04	73.05%
P51	104.50	93.95%	P87	91.66	86.43%	P51	24.30	98.86%	P25	7.04	75.10%
P74	93.51	94.45%	P74	88.89	87.48%	P42	22.88	99.06%	P23	7.04	77.16%
P15	90.47	94.94%	P20	87.23	88.50%	P39	20.67	99.24%	P85	6.16	78.95%
P87	89.17	95.41%	P19	82.05	89.47%	P37	14.86	99.37%	P24	6.16	80.74%
P37	87.88	95.88%	P5	78.55	90.39%	P78	13.51	99.49%	P81	6.16	82.54%
P45	87.57	96.35%	P45	78.00	91.31%	P89	11.44	99.59%	P54	6.16	84.33%
P20	85.53	96.81%	P15	76.58	92.21%	P21	11.14	99.69%	P31	6.16	86.13%

Table 3.12: (Table 3.11 continued) differential of optimized values and CPs for the indicated ID.

ID	CM (US$)	CP (%)	ID	VCM (US$)	CP (%)	ID	SC (US$)	CP (%)	ID	OC (US$)	CP (%)
P88	85.14	97.26%	P89	76.38	93.10%	P65	8.46	99.76%	P5	5.28	87.67%
P5	78.13	97.68%	P88	74.73	93.98%	P37	6.89	99.82%	P12	5.28	89.21%
P89	72.86	98.07%	P39	58.16	94.66%	P66	3.78	99.85%	P26	5.28	90.74%
P39	72.66	98.46%	P2	57.20	95.34%	P63	3.28	99.88%	P58	5.28	92.28%
P76	59.17	98.77%	P76	51.57	95.94%	P59	2.86	99.91%	P53	5.28	93.82%
P32	54.71	99.07%	P32	48.93	96.52%	P36	2.82	99.93%	P70	5.28	95.36%
P16	45.49	99.31%	P16	32.92	96.90%	P87	1.92	99.95%	P74	5.28	96.90%
P26	19.39	99.41%	P36	26.18	97.21%	P50	1.51	99.96%	P43	3.61	97.95%
P23	17.90	99.51%	P21	24.67	97.50%	P20	0.95	99.97%	P30	3.52	98.97%
P41	17.07	99.60%	P80	23.67	97.78%	P71	0.80	99.98%	P52	2.64	99.74%
P25	12.62	99.67%	P49	18.17	97.99%	P3	0.73	99.98%	P7	0.88	100.00%
P85	11.42	99.73%	P26	17.04	98.19%	P49	0.60	99.99%	P3	0.00	100.00%
P9	10.62	99.79%	P65	16.25	98.38%	P79	0.49	99.99%	P35	0.00	100.00%
P12	9.73	99.84%	P41	13.54	98.54%	P8	0.33	100.00%	P13	0.00	100.00%
P57	6.73	99.87%	P25	12.82	98.69%	P38	0.33	100.00%	P77	0.00	100.00%
P79	5.48	99.90%	P23	12.67	98.84%	P57	0.11	100.00%	P82	0.00	100.00%
P30	5.07	99.93%	P27	12.58	98.99%	P35	0.00	100.00%	P78	0.00	100.00%
P83	3.93	99.95%	P81	12.34	99.13%	P13	0.00	100.00%	P44	0.00	100.00%
P3	3.70	99.97%	P71	7.78	99.23%	P77	0.00	100.00%	P57	0.00	100.00%
P75	2.45	99.98%	P82	7.55	99.31%	P44	0.00	100.00%	P79	-0.88	0.05%
P7	1.57	99.99%	P66	7.24	99.40%	P7	-0.31	0.04%	P8	-0.88	0.11%
P52	0.84	100.00%	P57	6.62	99.48%	P74	-0.66	0.11%	P19	-0.91	0.16%
P24	0.38	100.00%	P85	6.58	99.55%	P60	-0.67	0.19%	P68	-1.76	0.27%
P49	0.30	100.00%	P12	5.91	99.62%	P88	-1.03	0.31%	P1	-2.64	0.43%
P35	0.00	100.00%	P79	5.87	99.69%	P76	-1.20	0.45%	P20	-2.64	0.59%
P13	0.00	100.00%	P37	5.47	99.76%	P32	-1.26	0.60%	P87	-4.40	0.86%
P77	0.00	100.00%	P30	4.40	99.81%	P85	-1.33	0.75%	P39	-6.16	1.23%
P44	0.00	100.00%	P9	4.30	99.86%	P12	-1.46	0.92%	P14	-7.92	1.71%
P65	-0.81	0.01%	P24	3.72	99.90%	P9	-1.60	1.11%	P72	-8.80	2.25%
P8	-1.62	0.03%	P3	2.97	99.94%	P75	-1.62	1.30%	P51	-13.20	3.05%
P71	-7.26	0.12%	P50	2.68	99.97%	P52	-1.80	1.51%	P89	-14.96	3.95%
P54	-13.30	0.29%	P1	1.01	99.98%	P23	-1.81	1.72%	P71	-15.84	4.92%
P21	-14.34	0.47%	P63	0.82	99.99%	P45	-1.87	1.94%	P15	-15.84	5.88%
P82	-16.59	0.68%	P83	0.39	100.00%	P31	-2.49	2.23%	P49	-18.48	7.00%
P37	-18.49	0.91%	P7	0.38	100.00%	P30	-2.85	2.57%	P16	-22.87	8.39%
P59	-18.82	1.15%	P52	0.00	100.00%	P26	-2.93	2.91%	P40	-24.63	9.88%
P31	-21.87	1.43%	P35	0.00	100.00%	P83	-4.38	3.42%	P22	-24.63	11.38%
P36	-22.03	1.71%	P13	0.00	100.00%	P46	-2.68	3.73%	P38	-24.79	12.89%
P50	-23.08	2.00%	P77	0.00	100.00%	P56	-3.39	4.13%	P65	-25.51	14.43%
P38	-24.62	2.31%	P44	0.00	100.00%	P5	-5.70	4.79%	P50	-27.27	16.09%
P66	-44.41	2.87%	P38	-0.16	0.00%	P55	-6.02	5.50%	P37	-30.79	17.96%
P27	-45.87	3.45%	P8	-0.41	0.01%	P54	-6.26	6.23%	P37	-30.85	19.83%
P81	-46.26	4.03%	P47	-3.27	0.05%	P41	-7.03	7.05%	P18	-31.58	21.75%
P1	-49.15	4.65%	P75	-3.85	0.11%	P25	-7.24	7.90%	P64	-36.95	23.99%
P47	-64.03	5.46%	P54	-13.20	0.30%	P61	-3.09	8.26%	P42	-41.35	26.50%
P55	-68.45	6.33%	P31	-25.54	0.66%	P24	-9.50	9.37%	P2	-45.75	29.28%
P63	-72.44	7.24%	P59	-30.48	1.09%	P43	-10.72	10.62%	P21	-50.15	32.33%
P43	-74.73	8.18%	P29	-34.01	1.58%	P68	-13.59	12.21%	P36	-51.03	35.43%
P29	-77.34	9.16%	P58	-39.25	2.13%	P70	-9.36	13.30%	P11	-55.43	38.79%
P80	-81.22	10.19%	P6	-49.38	2.83%	P34	-14.99	15.05%	P66	-55.43	42.16%
P56	-93.18	11.36%	P43	-67.62	3.79%	P82	-24.14	17.87%	P62	-59.83	45.79%
P6	-144.44	13.19%	P55	-70.35	4.79%	P6	-31.72	21.58%	P6	-63.34	49.63%
P58	-147.53	15.05%	P56	-101.23	6.23%	P1	-47.51	27.13%	P10	-73.08	54.07%
P70	-198.70	17.56%	P70	-194.63	9.00%	P29	-53.01	33.33%	P63	-76.54	58.72%
P34	-218.04	20.31%	P34	-214.48	12.04%	P81	-64.76	40.90%	P28	-81.82	63.69%
P53	-297.53	24.07%	P53	-234.11	15.37%	P27	-68.13	48.86%	P69	-86.22	68.92%
P60	-323.72	28.16%	P60	-331.73	20.08%	P47	-68.68	56.88%	P4	-96.60	74.79%
P46	-485.68	34.29%	P46	-493.55	27.09%	P53	-68.70	64.91%	P48	-97.66	80.72%
P68	-600.40	41.87%	P68	-585.05	35.40%	P73	-73.91	73.55%	P84	-98.54	86.70%
P61	-2107.64	68.48%	P61	-2114.81	65.44%	P80	-112.81	86.73%	P17	-102.93	92.95%
P73	-2496.07	100.00%	P73	-2433.27	100.00%	P58	-113.55	100.00%	P33	-116.13	100.00%
Total (saved)	10793.70		Total (saved)	1472.09		Total (saved)	10625.07		Total (saved)	-1303.86	

Table 3.13: ID of 89 products indicated in Table 3.8 with their equivalence in units.

P1 (1 unit = 1 bulk unit) American corn	P2 (1 unit = 1 kg) Apple	P3 (1 unit = 1 kg) Avocado
P4 (1 unit = 1 kg) Banana	P5 (1 unit = 1 pack of 1 kg) Bavarois dessert	P6 (1 unit = 1 kg) Beans
P7 (1 unit = 1 kg) Beans	P8 (1 unit = 1 kg) Beans (white)	P9 (1 unit = 1 pack of 1 kg) Beef puree
P10 (1 unit = 1 pack of 4 units) Beetroot	P11 (1 unit = 1 kg) Bird breast	P12 (1 unit = 1 pack of 1 kg) Bird puree
P13 (1 unit = 1 bulk unit) Bread bun	P14 (1 unit = 1 kg) Brisket	P15 (1 unit = 1 bulk unit) Broccoli
P16 (1 unit = 1 in bulk unit) Cabbage	P17 (1 unit = 1 kg) Carrot	P18 (1 unit = 1 in bulk unit) Celery
P19 (1 unit = 1 pack of 4 nits) Chard	P20 (1 unit = 1 kg) Cheese (gouda)	P21 (1 unit = 1 pack of 400 gr) Cheese flan
P22 (1 unit = 1 bulk unit) Cauliflower	P23 (1 unit = 1 pack of 1 kg) Corn starch	P24 (1 unit = 1 pack of 1 kg) Cream-asparagus
P25 (1 unit = 1 pack of 1 kg) Cream-vegetables	P26 (1 unit = 1 pack of 1 kg) Cream with no salt	P27 (1 unit = 1 pack of 1 kg) Caramel flan
P28 (1 unit = 1 in bulk unit) Cucumber salad	P29 (1 unit = 1 pack of 1 kg) Custard	P30 (1 unit = 1 pack of 1 kg) Delicacy
P31 (1 unit = 1 pack of 1 kg) Dried maize hominy	P32 (1 unit = 1 pack of 1000 cc) Fruit pulp	P33 (1 unit = 1 bulk unit) Egg
P34 (1 unit = 1 pack of 1 kg) Flour	P35 (1 unit = 1 bulk unit) French bread	P36(1 unit = 1 pack of 1 kg) Frozen corn
P37 (1 unit = 1 pack of 1 kg) Frozen peas	P38 (1 unit = 1 in bulk unit) Garlic	P39 (1 unit = 1 kg) Goose
P40 (1 unit = 1 kg) Grape	P41 (1 unit = 1 pack of 1 kg) Grits	P42 (1 unit = 1 kg) Ground beef
P43 (1 unit = 1 pack of 1 kg) Hair noodles	P44 (1 unit = 1 bulk unit) Hake fish	P45 (1 unit = 1 pack of 1 kg) Jelly
P46 (1 unit = 1 pack of 1000 ml) Lemon juice	P47 (1 unit = 1 pack of 1 kg) Lentils	P48 (1 unit = 1 in bulk unit) Lettuce
P49 (1 unit = 1 kg) Lima beans	P50 (1 unit = 1 pack of 1 kg) Margarine	P51 (1 unit = 1 bulk unit) Melon
P52 (1 unit = 1 pack of 1 kg) Meringue	P53 (1 unit = 1 pack of 1 kg) Milk flan	P54 (1 unit = 1 pack of 1 kg) Milk dessert (nevada)
P55 (1 unit = 1 pack of 1 kg) Milk pudding	P56 (1 unit = 1 pack of 1 kg) Milk TKF	P57 (1 unit = 1 kg) Mortadella (sausage)
P58 (1 unit = 1 pack of 1 kg) Mostaccioli (noodles)	P59 (1 unit = 1 pack of 1 kg) Mousse	P60 (1 unit = 1 pack of 1 kg) Oats
P61 (1 unit = 1 pack of 900 ml) Oil	P62 (1 unit = 1 kg) Orange	P63 (1 unit = 1 pack of 4 units) Parsley
P64 (1 unit = 1 kg) Peach	P65 (1 unit = 1 kg) Pear	P66 (1 unit = 1 pack of 4 kg) Pepper
P67 (1 unit = 1 kg) Plum	P68 (1 unit = 1 pack of 1 kg) Pork paste	P69 (1 unit = 1 kg) Potatoes
P70 (1 unit = 1 pack of 1 kg) Potato (inst mash)	P71 (1 unit = 1 kg) Prunes	P72 (1 unit = 1 kg) Pumpkin
P73 (1 unit = 1 pack of 1 kg) Rice	P74 (1 unit = 1 pack of 1 kg) Salsa-dessert	P75 (1 unit = 1 pack of 1 kg) Salt
P76 (1 unit = 1 pack 100 sachet) Salt (in sachet)	P77 (1 unit = 1 bulk unit) Sandwich bread	P78 (1 unit = 1 pack of 1 kg) Sausage
P79 (1 unit = 1 pack of 1 kg) Seafood (assortment)	P80 (1 unit = 1 pack of 400 gr) Spaghetti	P81 (1 unit = 1 pack of 1 kg) Spiral noodles
P82 (1 unit = 1 bulk unit) Squash (Italian)	P83 (1 unit = 1 kg) Sugar	P84 (1 unit = 1 kg) Tomato
P85 (1 unit = 1 pack of 1000 cc) Tomato puree	P86 (1 unit = 1 pack of 1 kg) Vegetables (frozen salad)	P87 (1 unit = 1 pack of 1 kg) Viennese
P88 (1 unit = 1 pack 1000 cc) Vinegar	P89 (1 unit = 1 bulk unit) Watermelon	

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