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Linear incidence rate: Its influence on the asymptotic behavior of a stochastic epidemic model

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Diseases are an important fact in the real world and concentrate the attention of a great number of researchers. Many of them are caused by nematodes, fungi, bacteria, or viruses. Nevertheless, there exists another, which are transmitted from the mothers to offspring (vertical transmission). In this paper, the dynamics of an suceptible-infectious (SI) epidemic model are analyzed considering a linear (bilinear or standard) incidence in the deterministic and stochastic regimes, assuming that the newborns are infected from their own mothers. A long-term behavior of the proportion of infected individuals depending on the system parameters and initial conditions is established. Then, we consider the case where this linear transmission rate, not previously used for this model, has a stochastic component described by a white noise which leads to a stochastic differential equation (SDE). The existence and uniqueness of the solution of the SDE is proved. The extinction of the disease is characterized, and an exponential decay to extinction is obtained under certain restrictions of the parameters. By assuming time-independent solutions of the Fokker-Plank equation, we determine a stationary measure of the probability density, and some of its properties are provided. Numerical simulations are performed to show the dynamics of the system in different regimes and to illustrate some differences between deterministic and stochastic effects.

KEYWORDS

epidemic model, linear incidence rates, stochastic transmission

MSC CLASSIFICATION

92B; 92C; 60H10

1 | INTRODUCTION

Currently, infectious diseases have taken increasing importance in the world, not only because of the damage to the people, but due to the high economic cost of treatment of the individuals, as also in their incidences in the development policies of the countries.[1] Many of the deterministic compartmental models have been described by ordinary differential equations (ODEs). But, the existence of factors that produce some uncertain effects on these models leads to consider randomness to properly describe the behavior of certain infectious diseases; as instance, taking stochastic disturbances into account in a deterministic model, generally the stochasticity can be incorporated in the form of Gaussian white noises.[2,3]

In this paper, a stochastic suceptible-infectious (SI) epidemic model, considering a bilinear or standard incidence rate is analyzed. Random factors are introduced, and a model described by a stochastic differential equation (SDE) is studied, considering a fatal disease in an animal population. The epidemic model is based on the model formulated in Roberts and Saha[4] in which the dynamics of bovine tuberculosis in possum populations in New Zealand was studied,[5] taking into account environmental variability in the form of Gaussian white noise in the disease transmission rate.[6-8] In that model,[4] vertical transmission[9] (mother to offspring) of the disease is assumed, as well as the contagion between individuals in the population. The bovine tuberculosis is a zoonotic illness produced by a bacteria named *Mycobacterium bovis* (*M. bovis*), related to the bacteria causative of the human and avian tuberculosis, but the model proposed in Roberts and Saha[4] can also be used to study other epidemics, such as AIDS transmission.[6,10] Other diseases, such as Hepatitis B, Syphilis, AIDS, Rubella, Varicella, Measles, Hepatitis, Poliomyelitis, and Mumps,[11] are one of the main transmission modes that infected mothers infect their unborn or newborn offsprings.[9]

About this paper, the stochastic model is based on a deterministic model described by an ODE, assuming the proportion of population infected varies with time, in which a term is added to account the effect of environmental fluctuations on the disease transmission rate and pseudo-vertical transmission, as it is assumed in Roberts and Saha.[4] To describe the epidemic more reasonably, we suppose the disease transmission rate is described by a linear (bilinear) incidence rate. If βC is the average number of enough contacts for the disease to be transmitted of a person or animal per unit time and $Z = I/N$ is the value of the number of infected, I, over the total population, N, then $\beta CI/N = \beta CZ$ is the average number of contacts with infectives per unit time of one susceptible (S), and $(\beta CI/N)S/N = \beta CZ(1-Z)$ is the proportion of new cases per unit time.[1] According to Ruan and Wang,[12] βCZ measures the infection force of the disease. If the proportion of the infectives in a population is very high, so that exposure to the disease agent is virtually certain, then the bilinear transmission rate is more sensitive than nonlinear rate to the increase in the number of infectives.[13] We think that this type of incidence rate is more flexible than that proposed by Roberts and Saha,[4] since it allows changes in the transmission rate according to the size of the infected proportion.

First, a complete description of the long-term behavior of the deterministic model is found (Theorem 3). Then, the asymptotic behavior of a stochastic fluctuation due to the environmental variation in the coefficient of disease incidence is studied. Thus, an SDE is obtained. The existence of its solution is proven along with establishing that it is unique and that its value is positive and less than one (Lemmas 4,5, and 6), allowing us to study the asymptotic behavior in the stochastic regime. All the values of the parameters for which the disease is extinguished are found in the stochastic model (Theorem 9). And in some cases, it is proven that the rate at which it is extinguished is of the exponential type (Theorems 7 and 8). The SDE is analyzed through the associated *Forward Kolmogorov equation or Fokker-Planck equation*[14] to obtain the probability density function[2] (its invariant probability distribution) when the proportion of the infected population reaches steady state. From this, we made a comparison between behaviors in long time of the deterministic and stochastic model (Table 2) that is exemplified via numerical simulation.

This paper is organized as follows: Section 2 presents the model under study. Section 3 assesses the deterministic model and the limit of proportion of population infected, Z_t, when time tends to infinity. Section 4 involves a random perturbation caused by environmental disturbance, which allows obtaining a SDE for Z_t. The existence and uniqueness of its solution is proved and that it remains in the interval (0,1). Section 5 is devoted of characterizing the long-time behavior of the stochastic process. Conditions under which the disease is extinguished are found out and the SDE is analyzed through an associated the Fokker-Plank equation to get an explicit expression for the probability distribution of the proportion of the population infected when Z_t reaches its steady state. Numerical simulations are performed to illustrate and compare the asymptotic behavior of the deterministic and stochastic models.

2 | DESCRIPTION OF THE MODEL

Assuming that N denotes the density of population per unit area and the individuals may be classified as either susceptible to infection (density S) or infected and infectious (density I), then $N = S+I$. Defining $Z(t) = \frac{I}{N}$ the proportion of the infected or infectious individuals, Roberts and Saha[4] consider an epidemiological model given by the system $X_\mu(N, Z)$ as follows:

$$\begin{cases} \frac{dN}{dt} = (B(N) - D(N) - \alpha Z)N \\ \frac{dZ}{dt} = -(1-p)B(N)Z + (\beta C(N) - \alpha)(1-Z)Z, \end{cases} \tag{1}$$

where $\mu = (\alpha, \beta, p) \in \mathbb{R}^3_+$ is the vector of parameter which have the following means:

α is the increase in mortality rate due to disease,
p is the probability of (pseudo) vertical transmission (mother to offspring) of disease (hence, $0 < p \leq 1$), and
β is the disease transmission rate.

The functions $B(N)$ and $D(N)$ represent the birth and death rates density-dependent of N, respectively, assuming that $B'(N) \leq 0$, $D'(N) \geq 0$ and $B(0) > D(0)$. Furthermore, $C(N)$ is the contact rate between individuals in the population. The term $\beta C(N)(1-Z)Z$ is the increase in the proportion of population that is infected or infective due to contact between individuals, $-\alpha Z$ corresponds to the decrease in the proportion of the population infected and infective due to death caused by the disease, and the term $-(1-p)B(N)Z$ is the decrease in the proportion of the population infected because of those who do not genetically inherit the disease.

Roberts and Saha[4] and Liu et al[8] assumed that $\frac{dN}{dt}$ tends to 0 much faster than $\frac{dZ}{dt}$, then the rates can be taken not dependent of the size of the population and that allow us to keep only the second equation in the system (1). Therefore, hereafter, only the second equation will be worked with.

Based on the model proposed in Roberts and Saha,[4] an extension is made considering the disease transmission rate depends on the proportion of population infected, that is, we consider a linear incidence rate β, given by

$$\beta(Z) = aZ \tag{2}$$

for $0 < Z \leq 1$ and the additional proportionality parameter $a > 0$, obtaining the model equation

$$\frac{dZ}{dt} = -(1-p)BZ + (aCZ - \alpha)(1-Z)Z. \tag{3}$$

We will consider this transmission rate β for populations where the proportion of infected population, Z, is positive, then we take $Z \in [Z^*, 1]$, with $0 < Z^* < 1$.

The rate $\beta(Z)$ changes fast or slow according to the value of the parameter a, depending on $a > 1$ or $a < 1$.

As far as we know, such rate has not been previously studied in this type of model and presents a more complex computation. The transmission rate (2) will be used for diseases with positive proportion infected population as initial condition. This type of rate is more flexible than the Roberts and Saha model (with constant rate) because it allows the transmission rate change according to the value of the proportion of infected individuals. We assume that the stochastic model here studied have different behavior in the long term than, the model analyzed in Roberts and Saha,[4] but also existing a fraction of infectious individuals under which the disease is extinguished.

3 | THE DETERMINISTIC MODEL AND THE LIMIT OF Z_t

In this section, we characterize the asymptotic behavior (when time elapses) of the process Z_t given by equation (3). After the following algebraic manipulation:

$$\begin{aligned}\frac{dZ}{dt} &= \left[-(1-p)B + (aCZ - \alpha)(1-Z)\right]Z \\ &= \left[-(1-p)B + aCZ - \alpha - aCZ^2 + \alpha Z\right]Z,\end{aligned}$$

and by regrouping properly we get the following:

$$\frac{dZ}{dt} = \left[-aCZ^2 + (\alpha + aC)Z - ((1-p)B + \alpha)\right]Z. \tag{4}$$

Defining

$$p(Z) = -CaZ^2 + (\alpha + Ca)Z - (B(1-p) + \alpha), \tag{5}$$

which is represented by a parabola open down, having two intercepts with the Z-axis when its discriminant Δ is positive, named

$$m = \frac{1}{2Ca}\left(\alpha + Ca - \sqrt{\Delta}\right) \quad \text{and} \quad K = \frac{1}{2Ca}\left(\alpha + Ca + \sqrt{\Delta}\right)$$

TABLE 1 Meanings of the parameters in equation (6)

Parameters	Meanings
aC	Growth rate of the proportion of the infectious individuals
K	Maximum size of the proportion of the infectious individuals
m	Extinction threshold of the proportion of the infectious individuals

with $0 < m < K$ and where

$$\Delta = (\alpha + Ca)^2 - 4Ca\,(B\,(1-p) + \alpha)\,. \tag{6}$$

Then, the quadratic polynomial $p(Z)$ can be written as

$$p(Z) = -Ca(Z-m)(Z-K) \quad \text{or} \quad p(Z) = Ca(m-Z)(Z-K).$$

Remark 1. Christen et al[15] and Gonzalez-Olivares[16] point out that if $p(Z)$ has two positive zeros with m,K, that is, $p(Z) = a\,(K-Z)\,(Z-m)$, then the equation

$$\frac{dZ}{dt} = aCZ\,(K-Z)\,(Z-m) \tag{7}$$

is similar to that one describes a simple Allee effect.[1617,18] The meaning of the parameter in the equation (6), equivalent to (4), are given in Table 1.

Hereafter, it will be defined by J the opposite of the constant term in the polynomial $p(Z)$,

$$J = (1-p)B + \alpha. \tag{8}$$

Note that $J \geq 0$ and is not related with $\beta(Z) = aZ$.

Since $\frac{dZ}{dt} = p(Z)Z$, it means $p(Z)$ is factor of $\frac{dZ}{dt}$ then the sign of $\frac{dZ}{dt}$ may be established through the sign of $p(Z)$; subsequently, we can have information on the growth or decline of the proportion of infected over time. To determine the limit of the infected proportion when time tends to infinity, the sign of $\frac{dZ}{dt}$ must be analyzed using Proposition 12 in Appendix A and equation (7). The results are presented in the following lemma and theorem. Lemma 1 establishes sufficient conditions on the parameters for the disease to be extinguished.

Lemma 1. *Given the epidemic model in (3) with linear incidence rate* $\beta(Z) = aZ$, $a > 0$. *The differential equation (3) can be rewritten as* $\frac{dZ}{dt} = p(Z)Z$, *where* $p(Z)$ *is given in expression (5) and* $0 \leq m \leq K$ *are the zeros of* $p(Z)$. *Then, for all initial condition* Z_0,

$$\lim_{t\to\infty} Z_t = 0,$$

if one of following conditions is verified:

(a) $\triangle \leq 0$
(b) $J > 0$, $\Delta > 0$ *and* $aCv < \alpha$.

Proof. In the case (a), we have $\frac{dZ}{dt} < 0$ (see Proposition 12) for all $0 < Z < 1$. We deduce that Z_t decreases to 0 for all $0 < Z < 1$. In a similar way, we obtain the same conclusion for the other cases. □

In the following theorem, we impose conditions over the parameters to get a complete description of the asymptotic behavior of the deterministic process.

Theorem 3. *Assume that the polynomial p(Z) described in (5) has real zeros $m \leq K$ and consider J defined in equation (8). Then, the asymptotic behavior of the proportion of infected population Z, solution of the epidemic model given by (3), and initial condition $Z_0 \in (0, 1)$ can be described by the following:*

1. *If $J = 0$, then $\alpha = 0$ y $p = 1$, therefore $m = 0$ y $K = 1$, then, when time passes all population is infected, that is,*

$$\lim_{t\to\infty} Z_t = 1.$$

2. *If $J > 0$ y $\Delta > 0$ and $aC > \alpha$, then $0 < m < K \leq 1$ and*

$$\lim_{t\to\infty} Z_t = \begin{cases} 0 \ \ si \ \ Z_0 \leq m \\ K \ \ si \ \ Z_0 > m. \end{cases} \tag{9}$$

Then, depending on the value of the proportion of infected population at the initial condition, the disease will die out or persist. Moreover, $K = 1$ if the offsprings always inherit the disease from their mothers, this is, if the probability of vertical transmission of the disease is 1. Therefore, when $K = 1$, all of the population become infected when the time elapses, if the proportion of the population infected exceeds the m.

3. *In other case, the disease is extinguished, that is,*

$$\lim_{t\to\infty} Z_t = 0.$$

Proof of Theorem 3. The demonstration is done studying the intervals of positivity and negativity of the derivative of Z with respect to time, t. Under the conditions of item 2, we analyze the sign of $\frac{dZ}{dt}$ in each interval. It follows that if $0 < Z_0 < m$, then Z_t decreases to 0, on the contrary, if $m < Z_0 < 1$ $\frac{dZ}{dt} > 0$, so in that interval, Z_t increases to the value K. The analysis is supported by the left panel of Figure 1, where it can see the curve of $\frac{dZ}{dt}$ as function of Z. In this case, the roots of $p(Z)$ are m and K, where $0 < m < K < 1$ and the parameters have values: $\alpha = 0.5$, $p = 0.9$, $C = 0.9$, $a = 0.95$ and $B = 0.2$, obtaining roots $m = 0.65$ and $K = 0.93$. We can see that the derivative $\frac{dZ}{dt}$ is negative between 0 and m, therefore the proportion of infected, Z, will be decreased to 0, and then Z decreases to 0 when the time elapses. If Z_0 is a value between m and K, $\frac{dZ}{dt}$ is positive in this interval then Z will be increasing to K. As $\frac{dZ}{dt} = 0$, the maximum of Z is reached in K, and moreover, this will be the limit of Z when the time goes by.

If $p = 1$ (all offsprings born infected), then $K = 1$, and when $Z_0 \in (m,1]$, all of the population is completely infected when time elapses. This situation is supported by the representation in the right panel of Figure 1 in which the zeros of the polynomial, $m = 0.58$ and $K = 1$, are such as $0 < m < K = 1$. The values of the parameters are $\alpha = 0.5$, $p = 1$, $C = 0.9$, $a = 0.95$, and $B = 0.2$. The graph of $\frac{dZ}{dt}$ has a similar shape to the previous case, but now if Z_0 is between m and K, the derivative of Z is positive then the proportion of infected, Z, will grow to 1, and therefore the whole population will be infected when the time passes.

If $J = 0$ (item 1), that is, $\alpha = 0$ and $p = 1$, then $m = 0$ y $K = 1$. It has, for any value of Z_0, $\frac{dZ}{dt} = aCZ^2(1-Z) > 0$, and over time, the entire population becomes infected, as seen in the left panel of Figure 2. There, the values of the parameters are $\alpha = 0$, $p = 1$, $C = 0.9$, $a = 0.8$, and $B = 0.6$. The dZ/dt curve takes positive values in the interval $[0, 1]$ and therefore the ratio of infected, Z, will grow to 1, and the whole population will be infected when time passes. Finally, with a similar

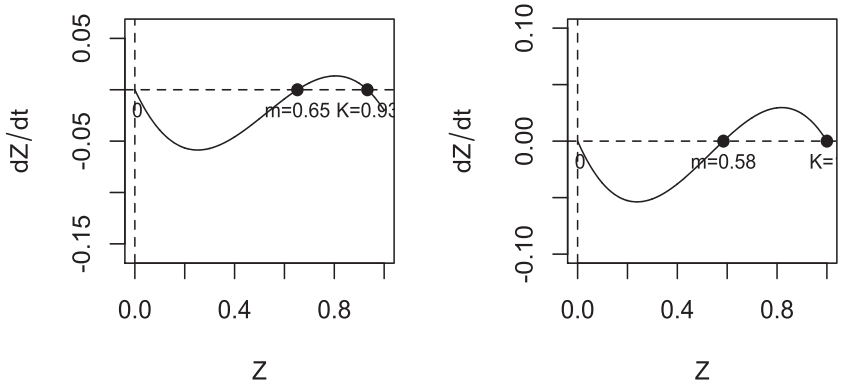

FIGURE 1 In the left panel, it shows the graph of $\frac{dZ}{dt}$ in function of Z when the roots of $p(Z)$ are $0 < m < K < 1$. The parameters have values: $\alpha = 0.5$, $p = 0.9$, $C = 0.9$, $a = 0.95$ and $B = 0.2$, obtaining roots $m = 0.65$ and $K = 0.93$. In the panel on the right, we can see the situation in which the zeros of the polynomial are of the type $0 < m < K = 1$. The values of the parameters are $\alpha = 0.5$, $p = 1$, $C = 0.9$, $a = 0.95$ and $B = 0.2$, obtaining roots $m = 0.58$ and $K = 1$

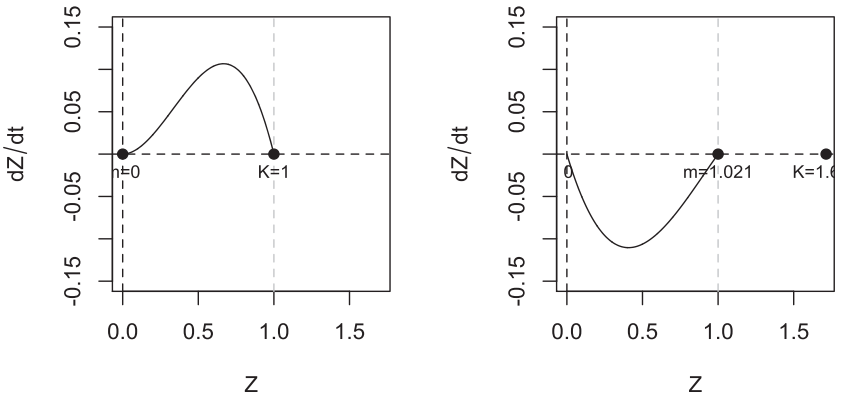

FIGURE 2 In the left panel of the figure, it can be observed that the situation in which the zeros of the polynomial are of the type $m = 0$ and $K = 1$ (with $\alpha = 0$, $p = 1$, $C = 0.9$, $a = 0.8$, and $B = 0.6$); on the contrary, in the right panel of the figure, it is shown that the graph of $\frac{dZ}{dt}$ in function of Z when the roots of $p(Z)$ are $0 < 1 < m < K$ and the values of the parameters are $\alpha = 0.6$, $p = 0.9$, $C = 0.7$, $a = 0.5$, and $B = 0.05$, obtaining roots $m = 1.02$ and $K = 1.69$

analysis, it follows that in any other situation, the disease is extinguished over time (see the right panel of the Figure 2 for an example). In this panel, the values $\alpha = 0.6$, $p = 0.9$, $C = 0.7$, $a = 0.5$, and $B = 0.05$ were used for the parameters. The dZ/dt curve takes negative values in the interval and therefore the ratio of infected, Z, will decrease to 0, and the the disease will be extinguished when time passes. □

In the Theorem 3, there are three possibilities when time tends to infinity, which the entire population is infected (items 1 and 2 when $p = 1$ and $Z_0 > m$), the proportion of infected population tends to a positive value K less than 1 (item 2 when $p < 1$ and $Z_0 > m$), that is, there is a persistence of the disease, or the disease is extinguished (item (2) when $Z_0 < m$ or under conditions in item 3). In the following, the presence of stochastic fluctuation in the model will be introduced, obtaining an SDE that will be studied. Finally, the asymptotic behavior of both models will be compared—deterministic and stochastic—with the aim of finding the choice of parameters for which this behavior could differ.

4 | SDE MODEL

The aim of this section is to include in the model the environmental variability as in Roberts and Saha,[4] Liu et al,[8] and Christen et al.[15] In these works, they assume that random factors affect the transmission of the disease. With this purpose, they add a coefficient to the disease transmission rate. In our case, we take into account random perturbation to the disease transmission coefficient a in such a way that β becomes

$$\beta(Z) = (a_0 + \rho\ \eta(t))Z = (a_0 Z) + \rho\ \eta(t)Z,$$

where $\eta(t)$ is a white noise and ρ is the environmental disturbance intensity. Then, equation (3) becomes as follows:

$$dZ = F(Z)dt + G(Z)\, dW_t, \tag{10}$$

where

$$F(Z) = -(1-p)\, BZ + (a_0 ZC - \alpha)(1-Z)\, Z \tag{11}$$

and

$$G(Z) = \rho C Z^2 (1-Z), \tag{12}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion. We are interested in proving that this SDE has a solution for all $t \geq t_0$, where t_0 is the initial time of the process, and that the solution remains in the interval $[0, 1]$. For that purpose, we prove Lemmas 4, 5, and 6.

Lemma 4. *Consider the autonomous SDE*

$$dZ = F(Z)dt + G(Z)\, dW_t, \quad t_0 \leq t < \infty,$$

where F(Z) and G(Z) are defined by (11) and (12), respectively. Then, there exists $K^* > 0$, $K^* = K^*(B, \alpha, a_0, C, \rho)$ such that the functions F(Z) and G(Z) satisfy the following global Lipschitz condition:

$$|F(x) - F(y)| + |G(x) - G(y)| \leq K^*|x - y|,$$

for any x, y∈[0,1].

Proof. We can bound both expressions $|F(x)-F(y)|$ and $|G(x)-G(y)|$ by

$$\begin{aligned}|F(x) - F(y)| &\leq B|(x - y)| + \alpha|(x - y)(1 - x - y)| + a_0C|x - y|\left|x + y - xy - y^2 - x^2\right| \\ &\leq |x - y|(B + 3\alpha + 5a_0C).\end{aligned}$$

For the second expression, we have

$$\begin{aligned}|G(x) - G(y)| &= \left|\rho C\left((1 - y)y^2 - (1 - x)x^2\right)\right| \\ &\leq 5\rho C|x - y|.\end{aligned}$$

Therefore, we define $K^* := \max\{B + 3\alpha + 5a_0C, 5\rho C\}$. □

Lemma 5. *The SDE considered in Lemma 4 subject to the initial condition $Z_{t_0} = Z_0$, where Z_0 is a random variable such that $0 < Z_0 < 1$ and $E[(Z_0)^2] < \infty$, admits a unique global solution for all t∈[t_0,∞).*

Proof. According to proposition 4.9 in Capasso and Bakstein[2] and the result obtained in Lemma 4, we conclude the claim. □

Lemma 6. *The solution of the model (10) with initial condition Z_0∈(0,1) will remain in this interval for all $t > t_0$.*

Proof. We follow closely the proof of theorem 3.1 in Gray et al.[7] For any initial condition $0 < Z_0 < 1$, by the previous lemmas, there exists a unique solution Z_t in (0,1) for t∈[0,τ_e), where $\tau_e = \inf\{t \geq 0 : Z_t \notin (0, 1)\}$ is the first time when Z_t leaves the interval (0,1).

After considering an integer $k_0 > 0$ such as $\frac{1}{k_0} < Z_0 < 1 - \frac{1}{k_0}$, we define an increasing sequence of stopping times $\tau_k = \inf\{t \in [0, \tau_e) : Z_t \notin \left(\frac{1}{k}, 1 - \frac{1}{k}\right)\}$ for $k \geq k_0$, where $\inf \varnothing = \infty$. We have $\tau_k \leq \tau_e$ for all k.

We will prove that $\tau_\infty := \lim_{k\to\infty}\tau_k = +\infty$ and thus $\tau_e = +\infty$. We will suppose that $\tau_e < +\infty$ and will arrive to a contradiction. If $\tau_e < +\infty$, then $\tau_\infty < \infty$ and there exist $T > 0$ and ε∈(0,1) such as $\mathbb{P}[\tau_\infty \leq T] > \varepsilon$. Since $(\tau_k < T) \supseteq (\tau_\infty \leq T)$ for all $k \geq k_0$, there exists $k_1 \geq k_0$ such as $\mathbb{P}[\tau_k \leq T] > \varepsilon$ for all $k \geq k_1$.

Let be a real function $V : (0, 1) \to \mathbb{R}$ defined by $V(X) = \frac{1}{x} + \frac{1}{1-x}$. Its derivatives are $V_x = -\frac{1}{x^2} + \frac{1}{(1-x)^2}$, $V_{xx} = 2\left(\frac{1}{x^3} + \frac{1}{(1-x)^3}\right)$.

Applying the Itô formula, we obtain

$$V(Z_t) = V(Z_0) + \int_0^t LV(Z_t)dt + \int_0^t \left(\frac{-1}{Z_t^2} + \frac{1}{(1 - Z_t)^2}\right) G(Z_t)dW_t, \tag{13}$$

for all $0 \leq t \leq T, k \geq k_1$, where $LV : (0, 1) \to \mathbb{R}$ is defined by $LV(x) = \frac{G^2(x)}{2}V_{xx}(x) + V_x(x)F(x)$. We can bound $LV(x)$ by

$$\begin{aligned}LV(x) &= 2\rho^2C^2x^4(1 - x)^2\left(\frac{1}{x^3} + \frac{1}{(1 - x)^3}\right) \\ &\quad + x\left(-aCx^2 + (aC + \alpha)x - J\right)\left(\frac{-1}{x^2} + \frac{1}{(1 - x)^2}\right) \\ &\leq V(X)[4\rho^2C^2 + aC] \leq \tilde{C}V(x),\end{aligned}$$

where $\tilde{C} = 4\rho^2C^2 + aC$.

Therefore, from (13),

$$\begin{aligned}\mathbb{E}(V_{t\wedge\tau_k}) &= V(Z_0) + \mathbb{E}\left[\int_0^{t\wedge\tau_k} LV(Z_s)ds\right] \\ &\leq V(Z_0) + \tilde{C}\mathbb{E}\left[\int_0^{t\wedge\tau_k} V(Z_s)ds\right] \\ &\leq V(Z_0) + \tilde{C}\mathbb{E}\left[\int_0^{t} V(Z_{s\wedge\tau_k})ds\right].\end{aligned} \tag{14}$$

Using the Gronwall inequality, we obtain

$$\mathbb{E}(V_{t\wedge\tau_k}) \leq V(Z_0 e \quad \tilde{C}T).$$

Define $\Omega_k = \{\tau_k \leq T\}$, for $k \geq k_1$, then $\mathbb{P}(\Omega_k) \geq \varepsilon$. For all w, $Z_{\tau_k} = \frac{1}{k}$, or $Z_{\tau_k} = 1 - \frac{1}{k}$, hence

$$V(Z_{\tau_k}) = k + \frac{k}{k-1}, \text{thus} V(Z_{\tau_k}) \geq k.$$

If $T\wedge\tau_k = \tau_k$, then $\forall w \in \Omega_k$, we have

$$k\varepsilon = k\mathbb{E}[I_{\Omega_k}(w)] \leq \mathbb{E}[I_{\Omega_k}(w)V(Z_{\tau_k})] \leq V(Z_0)e^{ct} \ < \ \infty.$$

Running $k \to \infty$, we reach a contradiction. □

We are now interested in determining the constraints on the parameters of the process to the disease will be extinguished almost sure (a.s.), that is, with a probability equal to one. This is the aim of the next section.

5 | ASYMPTOTIC BEHAVIOR ANALYSIS

In this section, we will analyze the behavior of the process when time tends to infinity. First, constraints on the parameters will be determined so that the disease is extinguished when time passes and then the asymptotic behavior of the process under the complementary situation will be studied. For this, the distribution of Z_t in the steady state will be found. The comparison of the long-time behavior of the deterministic and stochastic model is presented in Table 2, and some examples can be seen in Figures 3 to 7.

5.1 | Extinction

In the following theorems, we can find a value R_0^S (Theorem 7) for which the disease is extinguished if its value is less than 1. In fact, if $R_0^S < 1$, then $-(1-p)B + a_0C < 0$, which allows us to demonstrate the exponential decay of the proportion of infected to 0. The last inequation is interpreted as follows: the coefficient of the disease transmission rate times the contact rate does not exceed the product of the birth rate times the probability of not inheriting the disease. Another situation that ensures the disease dies out exponentially is presented in Theorem 8. Finally, all cases in which the disease will be extinguished (not necessarily exponentially) are characterized in the Theorem 9.

Theorem 7. *Let the stochastic epidemiological model (10), if:* $R_0^s = \frac{a_0C}{(1-p)B} < 1$ *for* $p < 1$, *then for initial condition* $Z_0 \in (0,1)$, *the solution of the process,* Z_t, *satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} log(Z_t) \leq a_0C - (1-p)B < 0, \ \textit{almost sure.}$$

Thus, Z_t *tends to 0 exponentially a. s., that is, the disease is extinguished with probability equal to 1.*

TABLE 2 Comparison of long-time behavior between deterministic and stochastic models

		Invariant measure ($P^*(Z)$, $p = 1$, $\alpha > 0$)				
Stochastic model		**Theorems 7,8, and 9 $p < 1$**	**$J = 0$ ($\alpha = 0$,$p = 1$) $\lim_{t\to\infty} Z_t = 1$**	**$a_0C+\rho^2C^2 < \alpha$ $\lim_{Z\to 0} P^*(Z) = +\infty$ $\lim_{Z\to 1} P^*(Z) = 0$**	**$a_0C+\rho^2C^2 = \alpha$ $\lim_{Z\to 0} P^*(Z) = +\infty$, $\lim_{Z\to 1} P^*(Z) = C_1$**	**$a_0C+\rho^2C^2 > \alpha$ $\lim_{Z\to 0} P^*(Z) = +\infty$, $\lim_{Z\to 1} P^*(Z) = +\infty$**
Deterministic model	$J = 0$	-		-	-	-
	($\alpha = 0$,$p = 1$)	-	Agree	-	-	-
	$\lim_{t\to\infty} Z_t = 1$	-	(Figure 5)	-	-	-
	$J > 0$,$\triangle > 0$,		-	-	-	-
	$a_0C > \alpha$	Agree	-	-	-	-
	$Z_0 < m$,	(Figure 3), LP	-	-	-	-
	$\lim_{t\to\infty} Z_t = 0$		-	-	-	-
	$J > 0$,$\triangle > 0$,		-	-	-	
	$a_0C > \alpha$	Disagree	-	-	-	Disagree
	$Z_0 > m$,	(Figure 3), RP	-	-	-	(Figure 7)
	$\lim_{t\to\infty} Z_t = K$		-	-	-	
	Other case	Agree	-	Agree	Disagree	Disagree
	$\lim_{t\to\infty} Z_t = 0$	(Figure 7)	-	(Figure 6), LP		(Figure 6), RP

Note. In this table, it is shows the different instances for the asymptotic behavior of the deterministic model in the rows of the stochastic model in the columns. In the cells, we can see the conclusion (if the results agree or disagree) and the figure number where a case was exemplified in the situations where numerical simulations were performed

Proof. We based on Gray et al,[7, Theorem4.1] by the Itô formula, we have $\log(Z_t) = \log(Z_0) + \int_0^t f(Z_s)ds + \int_0^t \rho C Z_u(1 - Z_u)dW_u$, where

$$\begin{aligned} f(x) &= -(1-p)B + (a_0Cx - \alpha)(1-x) - \frac{1}{2}\rho^2C^2x^2(1-x)^2 \\ &\leq -(1-p)B + (a_0C - \alpha) + \alpha x \\ &\leq -(1-p)B + a_0C, \end{aligned}$$

then $\log(Z_t) \leq \log(Z_0) + (-(1-p)B + a_0C)t + \rho C \int_0^t Z_u(1 - Z_u)dW_u$.

Applying limit when t tends to infinity,

$$\limsup_{t\to\infty} \frac{1}{t}\log(Z_t) \leq -(1-p)B + a_0C + \limsup_{t\to\infty} \frac{\rho C}{t}\int_0^t Z_u(1-Z_u)dW_u.$$

By the strong Large Number Law for local martingales (see Mao[19]), we have

$$\limsup_{t\to\infty} \frac{\rho C}{t}\int_0^t Z_u(1-Z_u)dW_u = 0,$$

therefore

$$\limsup_{t\to\infty} \frac{1}{t} log(Z_t) \leq -(1-p)B + a_0C < 0 \quad a.s.,$$

and we obtain our claim. □

Under the conditions required in Theorem 7, it is concluded that the disease is extinguished because the proportion of infected decreases, with an exponential rate of convergence, to 0, with a probability equal to one.

The conditions of Theorem 8 could have had intersection with those of the previous theorem in a certain situation but do not coincide and also ensure that the disease is extinguished exponentially.

Theorem 8. *If $a_0C \leq \alpha$ and $p < 1$ and $Z(0) = Z_0$ is the initial condition of the solution of the SDE (10), then*

$$\limsup_{t\to\infty} \frac{1}{t}\log(Z_t) = -(1-p)B < 0 a.s.,$$

then Z_t tends to 0 exponentially a.s., that is, the disease is extinguished with probability one.

Proof. We considerer $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = -J + (a_0C + \alpha)x - a_0Cx^2 - \frac{1}{2}\rho^2C^2x^2 + \rho^2C^2x^3 - \frac{1}{2}\rho^2C^2x^4$.

Since $-\frac{1}{2}\rho^2C^2x^2 + \rho^2C^2x^3 - \frac{1}{2}\rho^2C^2x^4 = -\frac{1}{2}\rho^2C^2x^2(1-x)^2 \leq 0$, then

$$\begin{aligned} f(x) &\leq -J + (a_0C + \alpha)x - a_0Cx^2 \\ &= -a_0C\left(x - \frac{a_0C + \alpha}{2a_0C}\right)^2 + a_0C\frac{(a_0C+\alpha)^2}{4a_0^2C^2}, \end{aligned}$$

that is, a quadratic polynomial in x. Set $g(x) = -J + (a_0C + \alpha)x - a_0Cx^2$, $g(x)$ is an increasing function in $x < \frac{a_0C+\alpha}{2a_0C}$. It is enough to show that g is increasing in $x \in [0, 1]$ and $g(1) < 0$. Computing $g(1) = -J + a_0C + \alpha - a_0C = -(1-p)B < 0$, if $p < 1$. Also, $\max_{0\leq x\leq 1} g(x) = g(1)$ is attained in $x_v = \frac{a_0C+\alpha}{2a_0C} \geq 1$ by the condition $a_0C \leq \alpha$. Therefore, reasoning as in previous theorems, we obtain

$$\limsup_{t\to\infty} \frac{log(Z_t)}{t} \leq \max_{0\leq x\leq 1} f(x) \leq g(1) < 0, \ a.s,$$

and we conclude the proof. □

Under the conditions of the Theorems 7 and 8, the disease is extinguished exponentially, this means that the extinction of the disease occurs at an exponential rate, of the type $e^{-((1-p)B-a_0C)t}$ and $e^{-((1-p)B)t}$, respectively, as time goes by Theorem 9 establishes which are broadest conditions for the disease to become extinct in the stochastic model.

Theorem 9. *Let the stochastic epidemiological model be $(Z_t)_{t\geq 0}$ given by (10), if the probability of vertical transmission satisfies $p < 1$, then for all initial condition $Z_0 \in (0,1)$, the process Z_t tends to 0 a.s., that is to say, the disease is extinguished with probability 1.*

Proof. Let $y_v = p(x_v)$ be the image of the vertex $x_v = \frac{1}{2} + \frac{\alpha}{2a_0C}$ of the quadratic function $p(Z)$ defined in (5) and set $y_1 = y_v + 2\rho^2C^2$.

Since $G(z)$ is a continuous function and increasing for $z \in (0, \frac{2}{3}]$ and decreasing for $z \in [\frac{2}{3}, 1)$, then $G(\xi) = \min_{x\in[\xi,1-\xi]}G(x)$. For $\lambda = \frac{2y_v}{G(\xi)}$, consider the real function $V(z)$ such that its derivative satisfies $V'(z) = \frac{e^{-\lambda z}}{G(z)}$ and $V(1/2) = 0$. Being that $V'(z) > 0$ for all $z \in (0,1)$, $V(z)$ is a bijective function in that interval and $\lim_{z\to 0^+}V(z) = -\infty$ and $\lim_{z\to 1^-}V(z) = \infty$. Applying the Itô formula, we obtain

$$V(Z_t) = V(Z_0) + \int_0^t LV(Z_t)dt + \int_0^t e^{-\lambda Z_u}dW_u, \tag{15}$$

for all $0 \leq t \leq T$, $k \geq k_1$, where $LV : (0,1) \to \mathbb{R}$ is defined by $LV(x) = \frac{G^2(x)}{2}V_{xx}(x) + V_x(x)F(x)$. We achieve

$$LV(x) = e^{-\lambda x}[\zeta(x) - (\lambda/2)G(x)],$$

where

$$\zeta(x) = \frac{-(1-p)B - \alpha + (a_0C + \alpha)x - a_0Cx^2}{\rho Cx(1-x)} + \frac{\rho C}{2}x(2-3x).$$

Since $\zeta(x)$ is a continuous function on (0,1) satisfying

$$\lim_{x\to 0^+}\zeta(x) = -\infty, \ \lim_{x\to 1^-}\zeta(x) = -\infty,$$

there exist two values $\xi \in (0, 5)$ and $r_1 < 0$ such that $\zeta(x) < r_1$ for $x \in (0,\xi)\cup(1-\xi, 1)$. For $z \in (0,\xi)\cup(1-\xi, 1)$, $LV(z) \leq r_1 - (\lambda/2)G(z) < 0$. For $z \in [\xi, 1-\xi]$, $LV(z) < y_1 - (\lambda/2)G(z) < 0$. Therefore, we can find $r_2 < 0$ such that $LV(z) < r_2$. We have

$$\frac{1}{t}V(Z_t) \leq \frac{1}{t}V(Z_0) + r_2 + \frac{1}{t}\int_0^t e^{-\lambda Z_u}dW_u. \tag{16}$$

By the strong Large Number Law for local martingales (see Mao[19]), we have

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t e^{-\lambda Z_t}dW_u = 0,$$

therefore

$$\limsup_{t\to\infty}\frac{1}{t}V(Z_t) = r_2 < 0 \quad a.s.$$

Since $V(z)$ is bijective and $\lim_{z\to 0^+}V(z) = -\infty$, we obtain our claim. □

We can see in Figure 3 the numerical simulation of 10 paths (grayscales lines) of the SDE joint with the solution of the deterministic process (dotted black line, see equation (3)) where the probability of vertical transmission, p, is such that $p < 1$, satisfying the conditions of Theorem 9. The parameter values chosen are $\alpha = 0.5$, $p = 0.9$, $C = 0.9$, $a_0 = 0.95$, $B = 0.2$, and $Z_0 = 0.5$ on the left panel. The parameters take the values $\alpha = 0.5$, $p = 0.9$, $C = 0.9$, $a_0 = 0.95$, $B = 0.35$, and $Z_0 = 0.8$ on the right panel. For both parameter choices, the conditions of Theorems 7 and 8 are not fulfilled since $R_0^S > 1$ and $a_0C > \alpha$, therefore we cannot conclude exponential decay. For this reason, we see that all paths of Z_t tend to zero (Theorem 9), but not necessarily with exponential rate.

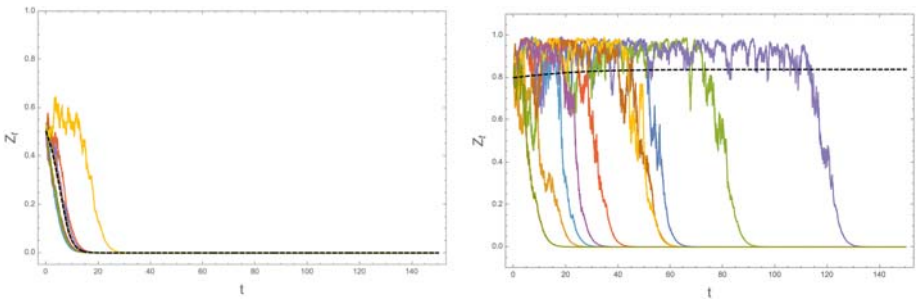

FIGURE 3 The left panel shows in grayscales 10 path of Z_t in terms of t and in dotted black line the solution of the deterministic model, when the parameters take the values $\alpha = 0.5$, $p = 0.9$, $C = 0.9$, $a_0 = 0.95$, $B = 0.2$, $\rho = 0.5$, and $Z_0 = 0.5$. The right panel shows both processes (lines in the same colors) also, but in this case, the parameters take the values $\alpha = 0.5$, $p = 0.9$, $C = 0.9$, $a_0 = 0.95$, $B = 0.35$, $\rho = 1$, and $Z_0 = 0.8$. In the figures, we can see that in both cases, the stochastic process dies out because $p < 1$, but for deterministic process, two situations are presented, corresponding to item 2 of Theorem 3. For the left panel, $Z_0 < m$, then the disease is extinguished agreeing with stochastic model, and for the right panel, $Z_0 > m$ then, Z_t tends to $K = 0.83$ the largest root of the polynomial $p(Z)$ [Colour figure can be viewed at wileyonlinelibrary.com]

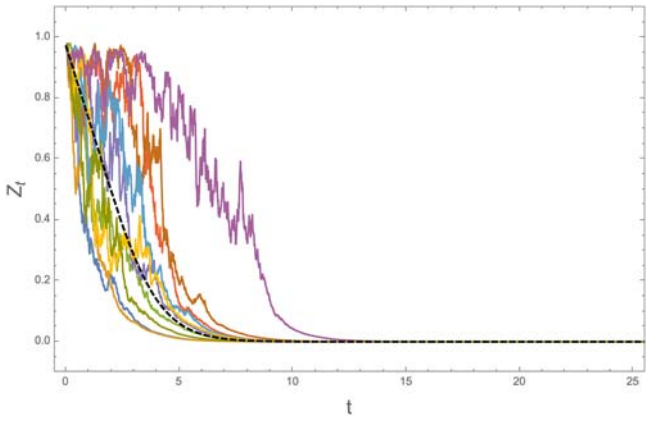

FIGURE 4 In the figure, it can be observed the 10 trajectories of the stochastic process (grayscales lines) and deterministic process (dotted black line) in the situation when the parameters values are $\alpha = 0.7$, $p = 0.6$, $C = 1$, $a_0 = 0.5$, $B = 0.6$, and $\rho = 1$ and the initial condition is $Z_0 = 0.97$. The deterministic process tends to 0 due to $a_0 C < \alpha$. Since $p < 1$, the trajectories of the stochastic process tends to 0, agreeing with the behavior of the deterministic process [Colour figure can be viewed at wileyonlinelibrary.com]

On the left panel, all the stochastic trajectories tend to reach the value of extinction of the disease, agreeing with the deterministic asymptotic behavior that corresponds to item 2 of Theorem 3, where for $Z_0 < m = 0.65$, the deterministic process tends to 0 when time goes by. On the right panel, all the stochastic trajectories tend also to the extinction of the disease, disagreeing with the deterministic asymptotic behavior (correspondent to item 2 of Theorem 3 but with initial condition $Z_0 > m = 0.65$) that in this case is approaching to the value of K what indicate that the disease will persist when time passes. We can see the dotted black line (deterministic process) that starts at $Z_0 = 0.8$ and tends to $K = 0.83$ when time elapses.

In Figure 4, it can be observed the situation when the parameters values are $\alpha = 0.7$, $p = 0.6$, $C = 1$, $a_0 = 0.5$, ad $B = 0.6$ and the initial condition is $Z_0 = 0.97$. The deterministic process tends to 0 due to $a_0 C < \alpha$. Since $p < 1$, the trajectories of the stochastic process tends to 0 (see Theorems 8, 7, and 9) and the disease becomes extinct, agreeing with the behavior of the deterministic process.

5.2 | Invariant measure

By using the Fokker-Planck equation,[5] associated with the stochastic equation under study, the asymptotic behavior of solutions of (10) can be investigated. This equation governs in its simplest version the probability function of the solution to a one-dimensional SDE (see Henderson and Plaschko[14]). The the Fokker-Planck equation associated to (10) is

$$\frac{\partial P(Z,t)}{\partial t} = -\frac{\partial}{\partial Z}(F(Z)P(Z,t)) + \frac{1}{2}\frac{\partial^2}{\partial Z^2}((G(Z))^2 P(Z,t)), \tag{17}$$

where $P(Z, t)$ is the probability density function of Z and $F(Z)$ and $G(Z)$ are defined by (11) and (12), respectively.

In order to find the probability density function $P^*(Z)$ when the proportion of infected population Z reaches the steady state, we analyze the solutions of equation (17) which equals 0 to the left side. In this case, $P(Z, t)$ becomes $P^*(Z)$, and it verifies

$$0 = -\frac{\partial}{\partial Z}(F(Z)P^*(Z)) + \frac{1}{2}\frac{\partial^2}{\partial Z^2}((G(Z))^2 P^*(Z)). \tag{18}$$

The unique solution for equation (18) under suitable conditions is given by

$$P^*(Z) = \frac{L}{(G(Z))^2} e^{\int_\epsilon^Z 2\frac{F(y)}{G^2(y)}dy},$$

where L is the normalizing constant, such that $\int_{Z_0}^1 P^*(Z)dZ = 1$ (see theorem 4.68 on Capasso and Bakstein[2] for more details).

By replacing $F(Z)$ and $G(Z)$, we obtain the following expression:

$$P^*(Z) = \frac{L}{\rho^2 C^2}(1-Z)^{-\frac{(2a_0C-6(1-p)B-2\alpha)}{\rho^2C^2}-2} Z^{\frac{(2a_0C-6(1-p)B-2\alpha)}{\rho^2C^2}-4} e^A, \tag{19}$$

where $A = \left(-\frac{1}{\rho^2C^2}\left(\frac{-J}{Z^2} + \frac{2a_0C-4(1-p)B-2\alpha}{Z} + \frac{2(1-p)BZ}{1-Z}\right)\right)$ and J is defined in equation (8).

We study the cases $p = 1$, $\alpha = 0$ and $p = 1$, $\alpha > 0$ in Propositions 10 and 11.

Proposition 10. *If $p = 1$ and $\alpha = 0$, the invariant measure is given by*

$$P^*(Z) = \frac{L}{\rho^2C^2}(1-Z)^{-\frac{2a_0C}{\rho^2C^2}-2} Z^{\frac{2a_0C}{\rho^2C^2}-4} e^{-\frac{2a_0C}{\rho^2C^2Z}} \tag{20}$$

and satisfies

$$\lim_{z\to 0^+} P^*(Z) = 0, \lim_{z\to 1^-} P^*(Z) = +\infty \text{ and } \int_0^1 P^*(Z)dZ = +\infty.$$

Proof. Since $e^{-\frac{2a_0C}{\rho^2C^2Z}}$ tends to 0 when $z\to 0^+$, then using L'Hospital rule, $\lim_{z\to 0^+} P^*(Z) = 0$. Since $-\frac{2a_0C}{\rho^2C^2} - 2 < 0$ for all values of the parameters $\{B,a_0,C,\rho\}$, $\lim_{z\to 1^-} P^*(Z) = +\infty$. Finally, since $\frac{2a_0C}{\rho^2C^2} + 2 \geq 1$, then for every choice of the values of the parameters $\{B,a_0,C,\rho\}$, $\int_0^1 P^*(Z)dZ = +\infty$. □

We can see in Figure 5 the numerical simulation of 10 paths (grayscales lines) of the SDE joint with the solution of the deterministic process (dotted black line, see equation (3)) where $p = 1$ and $\alpha = 0$, satisfying the conditions on Proposition 10. All the trajectories of stochastic process tends to reach the value of dissemination of the disease to the entire population agreeing with the deterministic asymptotic behavior that corresponds to item 1 of Theorem 3.

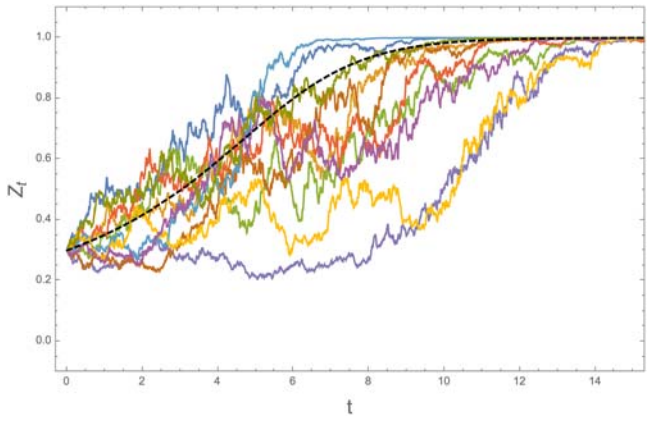

FIGURE 5 In the figure, it can be observed 10 trajectories of the stochastic process (grayscales lines) and the deterministic solution (dotted black line) in the situation in which the disease is diseminated to the entire population, when the zeros of the polynomial $p(Z)$ are $m = 0$ and $K = 1$ and the parameters take the following values: $\alpha = 0$, $p = 1$, $C = 0.9$, $a_0 = 0.8$, $B = 0.6$ and $\rho = 1$ and the initial condition is $Z_0 = 0.3$ [Colour figure can be viewed at wileyonlinelibrary.com]

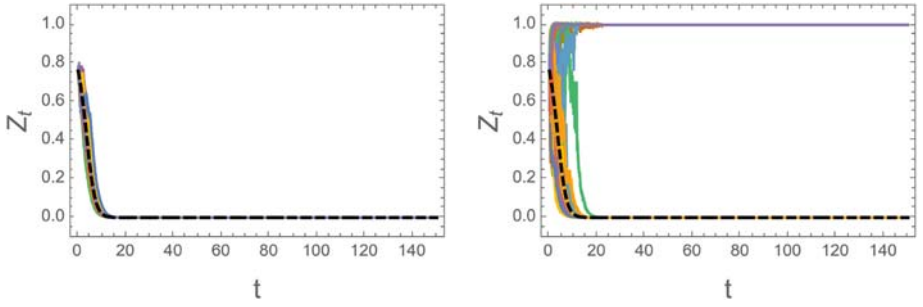

FIGURE 6 In the figure, it can be observed 50 trajectories of the stochastic process (grayscales lines) and the deterministic solution (dotted black line) in the situation when the parameters values are $\alpha = 0.6$, $p = 1$, $C = 0.3$, $a_0 = 1$, and $B = 0.2$ and the initial condition is $Z_0 = 0.76$. For both panels, the deterministic process tends to 0 due to $a_0C < \alpha$. In the left panel, $\rho = 1$, therefore $a_0C+\rho^2C^2 < \alpha$ what yields that the process tends to 0 for every path. In the right panel, $\rho = 5$, increasing the effect of environmental variability, and $a_0C+\rho^2C^2 > \alpha$, hence the trajectories tends to 0 and 1 and the disease may become extinct or spread to the entire population [Colour figure can be viewed at wileyonlinelibrary.com]

Proposition 11. *If $p = 1$ and $\alpha > 0$, the invariant measure is given*

$$P^*(Z) = \frac{L}{\rho^2C^2}(1-Z)^{-\frac{(2a_0C-2\alpha)}{\rho^2C^2}-2}Z^{\frac{(2a_0C-2\alpha)}{\rho^2C^2}-4}e^{-\frac{1}{\rho^2C^2}\left(\frac{-\alpha}{Z^2}+\frac{2a_0C-2\alpha}{Z}\right)} \tag{21}$$

and verifies the following properties:

(a) $\lim_{z\to 0^+} P^*(Z) = +\infty$.

(b) $$\lim_{z\to 1^-} P^*(Z) = \begin{cases} 0 & if\ \ a_0C+\rho^2C^2 < \alpha, \\ +\infty & if\ \ a_0C+\rho^2C^2 > \alpha, \\ C_1 & if\ \ a_0C+\rho^2C^2 = \alpha, \end{cases}$$
where $C_1 = \frac{L}{\rho^2C^2}e^{-\frac{1}{\rho^2C^2}(2a_0C-3\alpha)}$.

(c) Integrability in [0,1]: $\int_0^1 P^*(Z)dZ = +\infty$.

Proof. Since $e^{\frac{\alpha}{Z^2}}$ tends to $+\infty$ when $z\to 0^+$, then $\lim_{z\to 0^+} P^*(Z) = +\infty$. The convergence of $P^*(Z)$ when $z\to 1$ depends on the value of the exponent of $(1-Z)$, given the limits in item (b). In the neighborhood of $z = 1$, $P^*(Z)$ is integrable if $\frac{(2a_0C-2\alpha)}{\rho^2C^2} + 2 < 1$. In the neighborhood of $z = 0$, $P^*(Z)$ is not integrable, therefore $\int_0^1 P^*(Z)dZ = +\infty$. □

Numerical simulation of 50 paths (grayscales lines) of the SDE joint with the solution of the deterministic process (dotted black line, see equation (3)) are shown in Figures 6 and 7, where $p = 1$ and $\alpha > 0$, satisfying the conditions on Proposition 11, item (b). In both panels of Figure 6, the parameter values are $\alpha = 0.6$, $p = 1$, $C = 0.3$, $a_0 = 1$, and $B = 0.2$ and the initial condition is $Z_0 = 0.76$. In the left panel, $\rho = 1$, therefore $a_0C+\rho^2C^2 < \alpha$. The trajectories of stochastic process tend to reach 0, agreeing with the deterministic asymptotic behavior that corresponds to item 3 of Theorem 3. In the right panel of Figure 6, $\rho = 5$, therefore $a_0C+\rho^2C^2 > \alpha$. The deterministic process continues tending to 0, but the stochastic paths tend to both 0 and 1 due to the stochastic variability. Therefore, the behavior of both processes differs, since there are trajectories of the stochastic process that tend to 1, the whole population is infected with the disease, the opposite of the limit of the deterministic process, for which the disease is extinguished.

In Figure 7, the parameter values are $\alpha = 0.5$, $p = 1$, $C = 1$, $a_0 = 0.7$, $B = 0.6$, and $\rho = 1$ and the initial condition is $Z_0 = 0.72$. These values satisfy item 2 of Theorem 3 with $Z_0 > m = 0.71$, then the deterministic process tends to $K = 1$, infecting the entire population while for the stochastic process, $a_0C+\rho^2C^2 > \alpha$ (see (11), item (b)), then the trajectories tend to 0 or 1 due to the environmental fluctuation. Hence, the behavior of stochastic and deterministic model may disagree and in opposite behaviors because it can happen that according to the deterministic process, all of the population is infected while in the stochastic one the disease is extinguished.

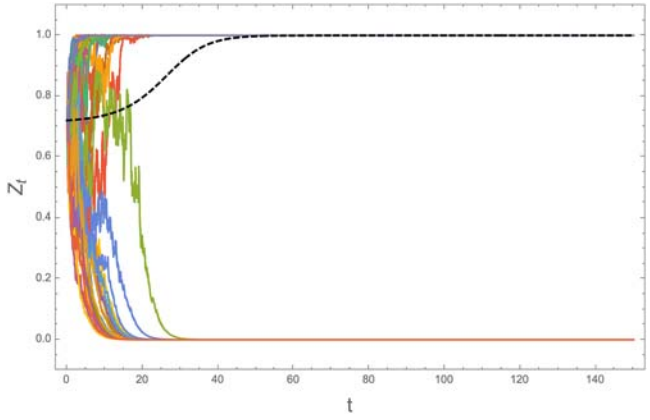

FIGURE 7 In the figure, it can be observed 50 trajectories of the stochastic process (grayscales lines) and the deterministic solution (dotted black line) in the situation when the parameters values are $\alpha = 0.5$, $p = 1$, $C = 1$, $a_0 = 0.7$, $B = 0.6$, and $\rho = 1$ and the initial condition is $Z_0 = 0.72$. The deterministic process tends to $K = 1$ due to $p = 1$ and $Z_0 > m = 0.71$ (see Theorem 3, item 2.). Since $a_0C+\rho^2C^2 > \alpha$, the path of stochastic process can tends to 0, extinction of the disease, or 1, the entire population is infected (see Proposition 11) [Colour figure can be viewed at wileyonlinelibrary.com]

For illustration purposes, we use the parameters involved in the simulation of Figure 7 to estimate, for large values of t, the proportion of trajectories that will tend to 0, $\hat{p}_0$, and the proportion of paths that will tend to 1, $\hat{p}_1$. Finally, to visualize a kind of expected value of the invariant measure, we compute $0 * \hat{p}_0 + 1 * \hat{p}_1 = \hat{p}_1$. The values $\hat{p}_0 = 0.40, 0.48, 0.45$, $\hat{p}_1 = 0.60, 0.52, 0.55$, and the "expected value" $\hat{p}_1 = 0.60, 0.52, 0.55$ are achieved from 50, 500, and 5.000 trajectories, respectively. Then, in about 45% of the trajectories, the disease will be extinguished, and in 55% of the trajectories, the whole population will be infected. Under the stochastic model, there is uncertainty about what will happen in the future.

6 | CONCLUSION

In this work, the influence of the bilinear incidence rate on the long-term behavior of a stochastic model was analyzed, considering an SI model. The illness model also takes into account vertical transmission of newborns, and the asymptotic behavior of the proportion of infected individuals was established. The stochastic variability was incorporated in the model as Gaussian white noise about the disease transmission rate. Our analytical analysis was reinforced with numerical simulations, and several results for the invariant measure were proved. For the first time, the model proposed in Roberts and Saha[4] is investigated with an incidence rate of bilinear type, allowing the use of the model to other diseases that characteristically have this type of rate.

In fact, we have provided a complete description of the solutions of the deterministic model (3), in function of its parameters, when $\frac{dZ}{dt}$ has positive real roots. These results characterize the limit of the proportion of infected when time tends to infinity for each of these solutions.

A stochastic model for the dynamics of the same infectious disease has been considered, and the existence and uniqueness of the solution in [0, 1] was demonstrated. The probability density distribution, $P^*(Z)$, when Z reaches the steady state was studied. That distribution doesn't integrate in [0, 1] because in $Z = 0$, it is divergent.

It was showed that the deterministic and stochastic cases can have different behaviors in the long term. The summary of the comparison of the long-term behavior of both processes (deterministic and stochastic) is presented in Table 2. We can see the conditions of Theorem 3 about deterministic behavior in the rows of the table and the conditions of stochastic behavior in the columns. Cells with dashes represent situations that are not possible. Cells with the word "figure" are those where a case of this type was represented in the corresponding figure. "LP" means left panel, and "RP" means right panel. Finally, all cases have been classified according to whether the behaviors agree or not.

As we can see in Table 2, the stochastic process will almost certainly be extinguished if the probability of vertical transmission of the disease is less than one ($p < 1$). For the deterministic process, when $p < 1$ there are cases in which the disease is extinguished and others in which it persists depending on the restrictions of the parameters and the initial condition Z_0.

When the vertical transmission of the disease occurs with certainty and the disease is not mortal, the entire population is infected when the time goes by both for the deterministic process and for the stochastic.

If the vertical transmission of the disease occurs with certainty and the disease has positive rate for mortality, the possible outcome will depend on the restriction $a_0C+\rho^2C^2$. It could be an extinction if this expression has a value minor than the positive rate for mortality, or extinguished or become all population infected for $a_0C+\rho^2C^2 > \alpha$. If $a_0C+\rho^2C^2 = \alpha$, the disease could persist with positive probability.

Different reasons can lead to the influence of environmental variability on the rate of disease transmission. If not taken into account, it can lead to erroneous conclusions about the behavior of a disease over time. In the model studied, it was shown that stochastic fluctuation can make the disease disappear, persist, or spread throughout the population, a different result from that obtained in the deterministic case.

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CONFLICTS OF INTEREST

This work does not have any conflicts of interest.

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APPENDIX A: CHARACTERIZATION OF THE ZEROS OF p(Z)

Proposition 12. *Let the polynomial and the discriminant given by the expressions (5) and (6), respectively, and J defined by (8). Then, the zeros of p(Z),*

$$m, K = \frac{-(aC+\alpha) \pm \sqrt{\triangle}}{-2aC}, \tag{A1}$$

are only nonnegative real numbers or complex numbers. The fully characterization of them is as follows:

1. *If $J = 0$, then $\alpha = 0$ y $p = 1$; therefore, $m = 0$ y $K = 1$.*
2. *If $J > 0$ and*

   (a) $\triangle > 0$, then

   (a.I) $m,K \in (0,1)$ if $aC > \alpha$ and $p < 1$.
   (a.II) $0 < m = \frac{\alpha}{aC} < K = 1$ if $aC > \alpha$ and $p = 1$.
   (a.III) $m = 1$ y $K \in (1,+\infty)$ if $\alpha > aC$ y $p = 1$.
   (a.IV) $m,K \in (1,+\infty)$ if $\alpha > aC$ y $p < 1$.

   (b) $\triangle = 0$, then

   (b.I) $0 < m = K < 1$ if $\alpha < aC$,
   (b.II) $0 < 1 < m = K$ if $\alpha > aC$,
   (b.III) $m = K = 1$ if $\alpha = aC$, and should be $p = 1$,

   (c) If $\triangle < 0$, then m and K are complex numbers. Therefor e, $p(Z) < 0$ for all Z.

Proof. The proof is immediate by applying Descartes' theorem and the formula of the roots of a quadratic polynomial. Write $p(Z) = b_2Z^2+b_1Z+b_0$ where $b_2 = -aC$, $b_1 = aC+\alpha$ and $b_0 = -(1-p)B-alpha$ then $b_2 < 0$, $b_1 > 0$ and $b_0 < 0$. Since $p(-Z) = b_2Z^2-b_1Z+b_0$ does not have sign changes a terms, using Descartes' theorem,[20] we conclude that $p(Z)$ does not have real zeros. Since $p(Z) = b_2Z^2+b_1Z+b_0$ has two sign changes (- to + and + to -), then by Descartes' theorem, the amount of positive zeros of $p(Z)$ is two or none. Then, the zeros of $p(Z)$ are only nonnegative real numbers or complex numbers.

If $J = 0$, since $\alpha \geq 0$ and $(1-p)B \geq 0$, then $\alpha = 0$ and $p = 1$. Under this restriction, the polynomial writes $p(Z) = -aCZ(Z-1)$ the $m = 0$ and $K = 1$. For the item $(a.I)$, we have that $\triangle > 0$, $aC-\alpha > 0$ and $p < 1$. Since $0 < m,K$, it is enough to demonstrate that the largest root is less than one, that is, $K < 1$. As $-(1-p)B \leq 0$ then $\triangle \leq (aC-\alpha)^2$, therefore $aC + \alpha + \sqrt{\triangle} < 2aC$. Hence, $K = \frac{(aC+\alpha)+\sqrt{\triangle}}{2aC} < 1$. The other items are proved in a similar way. □